

A RIEMANN-HURWITZ FORMULA FOR SKELETA IN NON-ARCHIMEDEAN GEOMETRY

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ABSTRACT. Let $\phi : C' \rightarrow C$ be a finite morphism between smooth, projective, irreducible curves defined over a non-archimedean valued, algebraically closed field k . The morphism ϕ induces a morphism between the analytifications of the curves. We will construct a compatible pair of deformation retractions of C'^{an} and C^{an} whose images $\Upsilon_{C'^{\text{an}}}$ and $\Upsilon_{C^{\text{an}}}$ are closed subspaces of C'^{an} and C^{an} which are homeomorphic to finite metric graphs (cf. Definition 2.3). We refer to such closed subspaces as skeleta. In addition, the subspaces $\Upsilon_{C'^{\text{an}}}$ and $\Upsilon_{C^{\text{an}}}$ are such that their complements in the two analytifications decompose into the disjoint union of Berkovich open balls and annuli. To these skeleta we can associate a genus. The pair of compatible deformation retractions forces the morphism ϕ to restrict to a map $\Upsilon_{C'^{\text{an}}} \rightarrow \Upsilon_{C^{\text{an}}}$. We will study how the genus of $\Upsilon_{C'^{\text{an}}}$ can be calculated using the morphism $\phi : \Upsilon_{C'^{\text{an}}} \rightarrow \Upsilon_{C^{\text{an}}}$.

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1. INTRODUCTION

Our goal in this paper is to study finite separable morphisms between smooth, projective, irreducible curves in terms of the maps they induce between certain closed subspaces of the curves which are homeomorphic to finite metric graphs (cf. Definition 2.3). We call such subspaces *skeleta*.

Let k be an algebraically closed, complete non-archimedean valued field. Let C' and C be smooth, projective, irreducible k -curves and $\phi : C' \rightarrow C$ be a finite morphism such that the extension of function fields $k(C) \hookrightarrow k(C')$ is separable. The morphism ϕ induces a morphism between the respective analytifications which we denote ϕ^{an} . Hence we have

$$\phi^{\text{an}} : C'^{\text{an}} \rightarrow C^{\text{an}}.$$

We prove that there exists a pair of 'compatible' deformation retractions:

$$\psi : [0, 1] \times C^{\text{an}} \rightarrow C^{\text{an}}$$

and

$$\psi' : [0, 1] \times C'^{\text{an}} \rightarrow C'^{\text{an}}$$

such that $\Upsilon_{C'^{\text{an}}} := \psi'(1, C'^{\text{an}})$ and $\Upsilon_{C^{\text{an}}} := \psi(1, C^{\text{an}})$ are skeleta in C'^{an} and C^{an} respectively with the following property. The analytic spaces $C'^{\text{an}} \setminus \Upsilon_{C'^{\text{an}}}$ and $C^{\text{an}} \setminus \Upsilon_{C^{\text{an}}}$ decompose into the disjoint union of Berkovich open disks and annuli. That is to say, there exist weak semi-stable vertex sets (cf. Definition 2.5) $\mathfrak{A} \subset C^{\text{an}}$ and $\mathfrak{A}' \subset C'^{\text{an}}$ such that $\Upsilon_{C^{\text{an}}} = \Gamma(\mathfrak{A}, C^{\text{an}})$ and $\Upsilon_{C'^{\text{an}}} = \Gamma(\mathfrak{A}', C'^{\text{an}})$.

The deformation retractions ψ and ψ' are said to be **compatible** if we have the following commutative diagram.

$$\begin{array}{ccc} [0, 1] \times C'^{\text{an}} & \xrightarrow{\psi'} & C'^{\text{an}} \\ \downarrow id \times \phi^{\text{an}} & & \downarrow \phi^{\text{an}} \\ [0, 1] \times C^{\text{an}} & \xrightarrow{\psi} & C^{\text{an}} \end{array}$$

As a consequence of the above diagram being commutative, we have the equality of sets $\Upsilon_{C'^{\text{an}}} = (\phi^{\text{an}})^{-1}(\Upsilon_{C^{\text{an}}})$.

In the case $C = \mathbb{P}_k^1$ and a morphism $\phi : C' \rightarrow \mathbb{P}_k^1$ where C' is a smooth, projective, irreducible k -curve, Hrushovski and Loeser have constructed a compatible pair of deformation retractions of the spaces $\widehat{C'}$ and $\widehat{\mathbb{P}_k^1}$ onto Γ -internal subspaces (Section 7, [HL]). For a k -variety V , the space \widehat{V} is very similar to the Berkovich space V^{an} (Section 13, [HL]). In section 2 we construct a compatible pair of deformation retractions given a finite, separable morphism $\phi : C' \rightarrow C$ by adapting the strategy employed in (Section 7, [HL]).

The morphism ϕ^{an} restricts to a surjective map $\Upsilon_{C'^{\text{an}}} \rightarrow \Upsilon_{C^{\text{an}}}$ between the skeleta. Since the spaces $\Upsilon_{C'^{\text{an}}}$ and $\Upsilon_{C^{\text{an}}}$ are homeomorphic to finite metric graphs, they can be decomposed into a set of vertices and a set of edges. Let $V(\Upsilon_{C^{\text{an}}})$ and $V(\Upsilon_{C'^{\text{an}}})$ denote the set of vertices of the skeleta $\Upsilon_{C^{\text{an}}}$ and $\Upsilon_{C'^{\text{an}}}$ respectively. We require that the pair $(V(\Upsilon_{C^{\text{an}}}), V(\Upsilon_{C'^{\text{an}}}))$ satisfies the following properties.

- (1) $V(\Upsilon_{C'^{\text{an}}}) = (\phi^{\text{an}})^{-1}(V(\Upsilon_{C^{\text{an}}}))$.
- (2) As described above, there exist weak semi-stable vertex sets $\mathfrak{A} \subset C^{\text{an}}$ and $\mathfrak{A}' \subset C'^{\text{an}}$ such that $\Upsilon_{C^{\text{an}}} = \Gamma(\mathfrak{A}, C^{\text{an}})$ and $\Upsilon_{C'^{\text{an}}} = \Gamma(\mathfrak{A}', C'^{\text{an}})$. We will require that $\mathfrak{A} \subset V(\Upsilon_{C^{\text{an}}})$ and $\mathfrak{A}' \subset V(\Upsilon_{C'^{\text{an}}})$.
- (3) If p (p') is a point on the skeleta $\Upsilon_{C^{\text{an}}}$ ($\Upsilon_{C'^{\text{an}}}$) for which there exists a sufficiently small open neighbourhood $U \subset \Upsilon_{C^{\text{an}}}$ ($U' \subset \Upsilon_{C'^{\text{an}}}$) such that $U \setminus \{p\}$

$(U \setminus \{p'\})$ has atleast three connected components then $p \in V(\Upsilon_{C^{\text{an}}})$ ($p' \in V(\Upsilon_{C'^{\text{an}}})$).

Likewise, let $E(\Upsilon_{C^{\text{an}}})$ and $E(\Upsilon_{C'^{\text{an}}})$ denote the set of edges of the skeleta $\Upsilon_{C^{\text{an}}}$ and $\Upsilon_{C'^{\text{an}}}$ respectively. We define the genus of the skeleton $\Upsilon_{C^{\text{an}}}$ as follows.

$$g(\Upsilon_{C^{\text{an}}}) = 1 - V(\Upsilon_{C^{\text{an}}}) + E(\Upsilon_{C^{\text{an}}}).$$

The morphism ϕ^{an} restricts to a morphism between the two skeleta. The genus of the skeleton $\Upsilon_{C'^{\text{an}}}$ can be calculated using invariants associated to the set of vertices $V(\Upsilon_{C'^{\text{an}}})$. To do so we define a divisor w on $\Upsilon_{C^{\text{an}}}$ whose degree is $2g(\Upsilon_{C'^{\text{an}}}) - 2$. A divisor on a finite metric graph is an element of the free abelian group generated by the points on the graph. For a point $p \in \Upsilon_{C^{\text{an}}}$, let $w(p)$ denote the order of the divisor at p . We define w as follows.

$$w(p) := \left(\sum_{e_p \in E_p, p' \in \phi^{-1}(p)} l(e_p, p') \right) - 2n_p.$$

We now define the terms in the expression above. For any $p' \in C'^{\text{an}}$ such that $\phi^{\text{an}}(p') = p$, the morphism ϕ^{an} induces a map between the tangent spaces at p' and p (cf. Definition 2.6). Let E_p be the set of germs (cf. Section 2.4) for which there exists a representative starting from p and contained completely in $\Upsilon_{C^{\text{an}}}$. Hence E_p is a subset of the tangent space at p . For every $p' \in (\phi^{\text{an}})^{-1}(p)$ and $e_p \in E_p$, let $l(e_p, p')$ denote the number of lifts (cf. Definition 3.1) of the germ e_p starting from p' . The cardinality of the set of preimages of the point p is denoted n_p . That is, $n_p := \#\{(\phi^{\text{an}})^{-1}(p)\}$.

In Section 5 we show that w is indeed a well defined divisor and that its degree is equal to $2g(\Upsilon_{C^{\text{an}}}) - 2$. We then study the variable n_p for $p \in C^{\text{an}}$ of type I and II and $l(e_p, p')$ described above when the morphism ϕ is Galois. These results are sketched below.

We assume only for section 5 that the morphism $\phi : C' \rightarrow C$ is such that the extension of function fields $k(C) \hookrightarrow k(C')$ is Galois. Let $p \in C^{\text{an}}$ of type I or II. Let s_p be the smallest real number belonging to the interval $[0, 1]$ such that $p \in \psi(s_p, C^{\text{an}})$. To calculate n_p we begin by defining an equivalence relation $\sim_{r(s_p)}$ on the set of k -points of the curve C' . Let $x'_1, x'_2 \in C'(k)$. We set $x'_1 \sim_{r(s_p)} x'_2$ if $\phi(x'_1) = \phi(x'_2)$ and $\psi'(s_p, x'_1) = \psi'(s_p, x'_2)$. Observe that each equivalence class is finite. For $x' \in C'(k)$, let $[x']_{r(s_p)}$ denote that equivalence class containing the point x' and $\#[x']_{r(s_p)}$ be its cardinality. In Lemma 5.2 we show that if $x'_1, x'_2 \in C'(k)$ are such that $\phi(x'_1) = \phi(x'_2)$ then

$$\#[x'_1]_{r(s_p)} = \#[x'_2]_{r(s_p)}.$$

Let $x \in C(k)$ and $x' \in \phi^{-1}(x)$. We set $r_{s_p}(x) := \#[x']_{r(s_p)}$. The lemma mentioned above implies that $r_{s_p}(\cdot)$ is well defined. By our assumption that the extension of function fields $k(C) \hookrightarrow k(C')$ is Galois, it follows that the ramification degree $\text{ram}(x'/x)$ is constant as x' varies along the set of preimages of x for the morphism ϕ . Let $\text{ram}(x) := \text{ram}(x'/x)$. For a point $p \in \Upsilon_{C^{\text{an}}}$, we use Proposition 5.3 to show that the value n_p can be calculated in terms of the invariants $r_{s_p}(\cdot)$ and $\text{ram}(\cdot)$ defined at the k -points of the curve C . More precisely, we prove that if $x \in C(k)$ is such that $\psi(s_p, x) = p$ then the following equality holds true.

$$n_p = [k(C') : k(C)] / (r_{s_p}(x) \cdot \text{ram}(x)).$$

If p is a point of type II, then the formula above simplifies further since the skeleton $\Upsilon_{C^{\text{an}}}$ contains every point belonging to $C(k)$ which is ramified. That is to say for every $x \in C(k)$ such that $\psi(s_p, x) = p$, $\text{ram}(x) = 1$.

We now proceed to simplifying the other term in the expression which defines the divisor w , namely $l(e_p, p')$ for $p' \in (\phi^{\text{an}})^{-1}(p)$ and $e_p \in E_p$. In Lemma 5.4 we

show that $l(e_p, p')$ remains constant as p' varies through the set of preimages $p' \in (\phi^{\text{an}})^{-1}(p)$. We set $l(e_p) := l(e_p, p')$. As explained before, the germ e_p is an element of the tangent space at p (cf. Definition 2.6). By Section 2.4.3, it corresponds to a discrete valuation of the \tilde{k} -function field $\widetilde{\mathcal{H}(p)}$. For any $p' \in (\phi^{\text{an}})^{-1}(p)$, the extension of fields $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$ can be decomposed into the composite of a purely inseparable extension and a Galois extension. Hence the ramification degree $\text{ram}(e'/e_p)$ is constant as e' varies through the set of lifts of the germ of e_p and starting at any $p' \in \phi^{-1}(p)$. Let $\text{ram}(e_p)$ be this number. In Proposition 4.5 we show that

$$l(e_p, p') = [k(C') : k(C)] / (n_p \text{ram}(e_p)).$$

We have thus shown that the genus of the skeleton $\Upsilon_{C'^{\text{an}}}$ can be computed in terms of the invariants $r_1(\cdot)$ defined at the k -points of the curve and $\text{ram}(\cdot)$ defined at the k -points of the curve and elements of the tangent space at a type II point of $\Upsilon_{C^{\text{an}}}$.

In section 4, we take another approach to understanding the genera of the two skeleta. We assume now that the morphism ϕ is separable but not necessarily Galois. It is known that

$$(1) \quad g(C) = g(\Upsilon_{C^{\text{an}}}) + \sum_{p \in V(\Upsilon_{C^{\text{an}}})} g_p.$$

where g_p is the genus of the \tilde{k} -function field $\widetilde{\mathcal{H}(p)}$ if p is of type II and zero otherwise. Likewise we have that

$$(2) \quad g(C') = g(\Upsilon_{C'^{\text{an}}}) + \sum_{p' \in V(\Upsilon_{C'^{\text{an}}})} g_{p'}.$$

Since the morphism $\phi : C' \rightarrow C$ is a finite separable morphism between smooth, projective curves, the Riemann-Hurwitz formula (Corollary IV.2.4, [H]) enables us to relate the genera of the curves C' and C . Precisely,

$$(3) \quad 2g(C') - 2 = \deg(\phi)(2g(C) - 2) + R$$

where R is a divisor on the curve C' such that if ϕ is tamely ramified at $x' \in C'$ then $\text{ord}_{x'}(R) = \text{ram}(x', x) - 1$. Using equations (1) and (2) above, we obtain the following formula relating the genera of the skeleta $g(\Upsilon_{C^{\text{an}}})$ and $g(\Upsilon_{C'^{\text{an}}})$.

$$2g(\Upsilon_{C'^{\text{an}}}) - 2 + 2(\sum_{p' \in V(\Upsilon_{C'^{\text{an}}})} g_{p'}) = \deg(\phi)(2g(\Upsilon_{C^{\text{an}}}) - 2) + \deg(\phi)(2\sum_{p \in V(\Upsilon_{C^{\text{an}}})} g_p) + R.$$

In the above equation the only vertices in $V(\Upsilon_{C^{\text{an}}})$ and $V(\Upsilon_{C'^{\text{an}}})$ for which the genera g_p or $g_{p'}$ are not necessarily zero are those of type II. Hence we restrict our attention to those vertices of type II. If $p' \in V(\Upsilon_{C'^{\text{an}}})$ then $\phi^{\text{an}}(p') = p$. The point $p \in C^{\text{an}}$ corresponds to a multiplicative norm on the function field $k(C)$ and the collection of norms on $k(C')$ which extend p are precisely the set of preimages $(\phi^{\text{an}})^{-1}(p)$. The extension of non-archimedean valued complete fields $\mathcal{H}(p) \hookrightarrow \mathcal{H}(p')$ induces an extension of \tilde{k} -function fields $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$. The \tilde{k} -function fields $\widetilde{\mathcal{H}(p)}$ and $\widetilde{\mathcal{H}(p')}$ both correspond to smooth projective \tilde{k} curves which we denote by \tilde{C}_p and $\tilde{C}'_{p'}$ respectively. If the morphism $\tilde{C}'_{p'} \rightarrow \tilde{C}_p$ was separable then the Riemann-Hurwitz formula relates the numbers $g_{p'}$ and g_p .

Let $I(p', p)$ denote the subfield of $\widetilde{\mathcal{H}(p')}$ which contains $\widetilde{\mathcal{H}(p)}$ such that $I(p', p)$ is purely inseparable over $\widetilde{\mathcal{H}(p)}$ of degree $i(p)$ and $I(p, p') \hookrightarrow \widetilde{\mathcal{H}(p)}$ is a separable field extension of degree $i(p')$. The function field $I(p', p)$ corresponds to a smooth projective \tilde{k} -curve which is homeomorphic to \tilde{C}_p . We denote this curve $W_{p', p}$. Since $I(p', p)$ is purely inseparable over the function field $\widetilde{\mathcal{H}(p)}$ the genus of the

curve $W_{p',p}$ is equal to g_p . Applying the Riemann-Hurwitz formula to the extension $I(p',p) \hookrightarrow \widehat{\mathcal{H}(p')}$ and substituting in the equation above gives

$$2g(\Upsilon_{C'^{\text{an}}}) - 2 = \deg(\phi)(2g(\Upsilon_{C^{\text{an}}}) - 2) + \sum_{p \in V(\Upsilon_{C^{\text{an}}})} A_p(2g_p - 2) + \sum_{p \in V(\Upsilon_{C^{\text{an}}})} 2(n_p + \deg(\phi)) + R + \sum_{p' \in V(\Upsilon_{C'^{\text{an}}})} R_{p'}.$$

The constant A_p is defined to be $[\sum_{p' \in (\phi^{\text{an}})^{-1}(p)} i(p')] + \deg(\phi)$. The number of preimages of a point $p \in C^{\text{an}}$ for the morphism ϕ^{an} is denoted n_p . The ramification divisor associated to the Riemann-Hurwitz formula for the field extension $I(p',p) \hookrightarrow \widehat{\mathcal{H}(p')}$ is denoted $R_{p'}$.

The remainder of Section 4 is dedicated to simplifying some of the terms that appear in the equation above.

2. PRELIMINARIES

2.1. The analytification of a k -variety. In this section we discuss the analytification of a smooth, projective curve over the non-archimedean valued, algebraically closed field k .

Let X be a reduced, separated variety of finite type over the field k . Associated functorially to X is a Berkovich analytic space X^{an} . We examine this notion in more detail.

Let $k\text{-an}$ denote the category of k -analytic spaces (1.2.4, [B2]), Set denote the category of sets and $\text{Sch}_{\text{lft}/k}$ denote the category of schemes which are locally of finite type over k . We define a functor

$$F : k\text{-an} \rightarrow \text{Set} \\ Y \mapsto \text{Hom}(Y, X)$$

where $\text{Hom}(Y, X)$ is the set of morphisms of k -ringed spaces. The following theorem defines the space X^{an} .

Theorem 2.1. (3.4.1, [B]) *The functor F is representable by a k -analytic space X^{an} and a morphism $\pi : X^{\text{an}} \rightarrow X$. For any non-archimedean field K extending k , there is a bijection $X^{\text{an}}(K) \rightarrow X(K)$. Furthermore, the map π is surjective.*

The associated k -analytic space X^{an} is *good*. This means that for every point $x \in X^{\text{an}}$ there exists a neighbourhood of x isomorphic to an affinoid space. Theorem 2.1 implies the existence of a well defined functor

$$(\)^{\text{an}} : \text{Sch}_{\text{lft}/k} \rightarrow \text{good } k\text{-an} \\ X \mapsto X^{\text{an}}.$$

We follow (Section 4, [AB]) to describe X^{an} . As a set X^{an} is the collection of pairs $\{(x, \eta)\}$ where x is a scheme theoretic point of X and η is a valuation on the residue field $k(x)$ which extends the valuation on the field k . We endow this set with a topology as follows. A pre-basic open set is of the form $\{(x, \eta) \in U^{\text{an}} \mid |f(\eta)| \in W\}$ where U is a Zariski open subset of X with $f \in \mathcal{O}_X(U)$, W an open subspace of $\mathbb{R}_{\geq 0}$ and $|f(\eta)|$ is the evaluation of the image of f in the residue field $k(x)$ at η . A basic open set is any set which is equal to the intersection of a finite number of pre-basic open sets. Properties of the scheme translate to properties of the associated analytic space. If X is proper then X^{an} is compact and if X is connected then X^{an} is pathwise connected.

Let C be a smooth, projective, irreducible k -curve. We divide the points of C^{an} into four groups using the description above. For a point $\mathbf{x} := (x, \mu) \in C^{\text{an}}$, let $\mathcal{H}(x)$ denote the completion of the residue field $k(x)$ for the valuation η . Let $t(\mathbf{x})$ denote the transcendence degree of the residue field $\widehat{\mathcal{H}(\mathbf{x})}$ over \tilde{k} and $d(\mathbf{x})$ the rank

of the group $|\mathcal{H}(\mathbf{x})^*|/|k^*|$. Abhyankar's inequality implies that $t(\mathbf{x}) + d(\mathbf{x}) \leq 1$. This allows us to classify points. We call \mathbf{x} a type I point if it is a k -point of the curve. In which case, both $t(\mathbf{x})$ and $d(\mathbf{x})$ will be zero. If $t(\mathbf{x}) = 1$, then $d(\mathbf{x}) = 0$ and we will say that such a point is of type II. If $d(\mathbf{x}) = 1$ then $t(\mathbf{x}) = 0$, and we will say that such a point is of type III. Lastly, if $t(\mathbf{x}) = d(\mathbf{x}) = 0$, and \mathbf{x} is not a k -point of the curve, we will say that \mathbf{x} is of type IV.

The fact that C is smooth, projective and irreducible implies that the analytification C^{an} is Hausdorff, compact and pathwise connected. As an example we will describe the analytification of the projective line $\mathbb{P}_k^{1,\text{an}}$.

2.2. $\mathbb{P}_k^{1,\text{an}}$ - The analytification of the projective line over k .

2.2.1. Types of points: The set of type I points are the k -points $\mathbb{P}_k^1(k)$ of the projective line. The type II, III and IV points are of the form (ζ, μ) where ζ is the generic point of \mathbb{P}_k^1 and μ is a multiplicative norm on the function field $k(\mathbb{P}_k^1)$ which extends the valuation on the field k . The field $k(\mathbb{P}_k^1)$ is of the form $k(T)$. Hence describing the set of points of $\mathbb{P}_k^{1,\text{an}} \setminus \mathbb{P}_k^1(k)$ is equivalent to describing the set of multiplicative norms on the function field $k(T)$ which extend the valuation on k . Let $a \in \mathbb{P}_k^1(k)$ be a k -point and $B(a, r) \subset k$ denote the closed disk around a of radius r contained in $\mathbb{P}_k^1(k)$. We define a multiplicative norm $\eta_{a,r}$ on $k(T)$ as follows. Let $f \in k(T)$. We set $|f(\eta_{a,r})| := \max_{y \in B(a,r)} |f(y)|$. It can be checked that this is a multiplicative norm on the function field. If r belongs to $|k^*|$ then $(\zeta, \eta_{a,r})$ is a type II point. Otherwise $(\zeta, \eta_{a,r})$ defines a type III point. It can be shown that every type II and type III point is of this form. A type IV point corresponds to a decreasing sequence of closed disks with empty intersection. Let J be a directed index set and for every $j \in J$, $B(a_j, r_j)$ be a closed disk around $a_j \in k$ of radius r_j such that $\bigcap_{j \in J} B(a_j, r_j) = \emptyset$. Let $\mathfrak{E} := \{B(a_j, r_j) | j \in J\}$. We define a multiplicative norm $\eta_{\mathfrak{E}}$ on the function field as follows. For $f \in k(T)$, let $|f(\eta_{\mathfrak{E}})| := \inf_{j \in J} \{\max_{y \in B(a_j, r_j)} |f(y)|\}$. The set of multiplicative norms on $k(T)$ defined in this manner corresponds to the set of type IV points in $\mathbb{P}_k^{1,\text{an}}$.

It is standard practice to describe the points of $\mathbb{A}_k^{1,\text{an}}$ as the collection of multiplicative seminorms on the algebra $k[T]$ which extend the valuation of the field k . As a set $\mathbb{P}_k^{1,\text{an}} = \mathbb{A}_k^{1,\text{an}} \cup \{\infty\}$ where $\infty \in \mathbb{P}_k^1(k)$ is the complement of the affine subspace $\text{Spec}(k[T]) \subset \mathbb{P}_k^1$.

2.2.2. The Berkovich Closed and Open disks/Open and Closed annuli. The topological space $\mathbb{P}_k^{1,\text{an}}$ is compact, simply connected and Hausdorff. We now proceed to describe certain subspaces of $\mathbb{A}_k^{1,\text{an}} \subset \mathbb{P}_k^{1,\text{an}}$. Let $a \in k$. The Berkovich open ball around a of radius r is denoted $\mathbf{O}(a, r)$ and defined by the following equation

$$\mathbf{O}(a, r) := \{p \in \mathbf{A}_k^{1,\text{an}} | |(T - a)(p)| < r\}.$$

Similarly, the Berkovich closed disk around a of radius r is denoted $\mathbf{B}(a, r)$ and defined by the following equation

$$\mathbf{B}(a, r) := \{p \in \mathbf{A}_k^{1,\text{an}} | |(T - a)(p)| \leq r\}.$$

In fact $\mathbf{B}(a, r)$ is the set of bounded, multiplicative semi-norms on the affinoid algebra $k\{r^{-1}T\}$ which extends the valuation on the field k . Hence, we have $\mathbf{B}(a, r) = \mathcal{M}(k\{r^{-1}(T - a)\})$ (1.4.4, [B]) where $\mathcal{M}(k\{r^{-1}(T - a)\})$ denotes the spectrum of the Banach algebra $k\{r^{-1}(T - a)\}$. The spectrum of a Banach algebra and the concept of an affinoid algebra are discussed in detail in (1, [B]). The collection of open sets which are either Berkovich open disks or the complement of a Berkovich closed disk forms an open pre-basis for the topology on $\mathbb{P}_k^{1,\text{an}}$.

The punctured open disk around a of radius r is the space $\mathbf{O}(a, r) \setminus \{a\}$. We now define the open and closed annulus. An open annulus $\mathcal{A}(a; r, R)^-$ is of the form $\{x \in \mathbb{A}_k^{1, \text{an}} \mid r < |(T - a)(x)| < R\}$ where $r < R$ are a pair of positive real numbers. Similarly, $\mathcal{A}(a; r, R)$ is a closed annulus if it is of the form $\{x \in \mathbb{A}_k^{1, \text{an}} \mid r \leq |(T - a)(x)| \leq R\}$ for any pair of positive real numbers $r \leq R$. The closed annulus $\mathcal{A}(a; r, R)$ is the spectrum of the affinoid algebra $k\{R^{-1}T, rT^{-1}\}$.

The *skeleton* of an open annulus or a punctured Berkovich open disk will be of use when we associate to the semi-stable vertex set of a smooth, projective curve, a closed subspace of the analytification of the curve which has the structure of a connected, finite metric graph and onto which the analytification of the curve retracts. The **skeleton of the open annulus** $\mathcal{A}(a; r, R)^-$ is the set $\{\eta_{a,t} \mid r < t < R\}$ (cf. 2.2.1) equipped with the induced topology. It is denoted $S_{\mathcal{A}(a; r, R)^-}$. The map

$$\begin{aligned} S_{\mathcal{A}(a; r, R)^-} &\rightarrow (\log(r), \log(R)) \\ \eta_{a,t} &\mapsto \log(t) \end{aligned}$$

is a homeomorphism onto the real interval $(\log(r), \log(R))$ and we identify $S_{\mathcal{A}(a; r, R)^-}$ with $(\log(r), \log(R))$ using this homeomorphism. Similarly, the **skeleton of the punctured Berkovich open disk** $\mathbf{O}(a, r) \setminus \{a\}$ is the set of points $\{\eta_{a,t} \mid 0 < t < r\}$ provided with the induced topology. It is denoted $S_{\mathbf{O}(a, r) \setminus a}$. The map

$$\begin{aligned} S_{\mathbf{O}(a, r) \setminus a} &\rightarrow (-\infty, \log(r)) \\ \eta_{a,t} &\mapsto \log(t) \end{aligned}$$

is a homeomorphism and we use it to identify $S_{\mathbf{O}(a, r) \setminus a}$ with the real interval $(-\infty, \log(r))$.

2.3. Semi-stable vertex sets. We introduce the notion of a semi-stable vertex set of a smooth, projective curve and the skeleton associated to it. What follows in this section is a restatement of (4.4, [AB]).

Let C be a smooth, projective, irreducible curve defined over the field k and C^{an} be its analytification.

Definition 2.2. A *semi-stable vertex set* \mathfrak{V} for C^{an} is a finite collection of points of type II such that the complement of this set in C^{an} decomposes into the disjoint union of open subspaces each isomorphic either to a Berkovich open disk or an open annulus. Such a decomposition of the space $C^{\text{an}} \setminus \mathfrak{V}$ is called a *semi-stable decomposition*.

Definition 2.3. An *abstract finite metric graph* comprises of the following data: A finite set of *vertices* W , a set of *edges* $E \subset W \times W$ which is symmetric and a function $l : E \rightarrow \mathbb{R}_{>0} \cup \infty$ such that if $(x, y) \in E$ then $l(x, y) = l(y, x)$.

The function l is called the length function. We associate a metric space to an abstract finite metric graph by the following rule. Any two vertices v_1 and v_2 such that $(v_1, v_2) \in E$ are joined by a real interval of length $l(v_1, v_2)$. A *finite metric graph* is the space associated as described above to an abstract finite metric graph.

Definition 2.4. The *skeleton associated to a semi-stable vertex set* \mathfrak{V} is defined to be the union of the skeleta of all open annuli which occur in the semi-stable decomposition along with the vertex set \mathfrak{V} . It is denoted $\Gamma(C^{\text{an}}, \mathfrak{V})$.

The space C^{an} is pathwise connected. Hence for any two points $v_1, v_2 \in \mathfrak{V}$, there exists a path between them. From the nature of the semi-stable decomposition, each such path must be the union of a finite number of edges of the skeleton $\Gamma(C^{\text{an}}, \mathfrak{V})$. It follows that $\Gamma(C^{\text{an}}, \mathfrak{V})$ is pathwise connected. The skeleton of every open annulus

which occurs in the semi-stable decomposition is equipped with a length function (cf. Section 2.2.2). This defines a metric on $\Gamma(C^{\text{an}}, \mathfrak{V})$. We have thus equipped the skeleton $\Gamma(C^{\text{an}}, \mathfrak{V})$ with the structure of a finite, metric graph.

The semi-stable reduction theorem guarantees the existence of a semi-stable vertex set and any finite set of type II points can be enlarged to become a semi-stable vertex set. A theorem of Berkovich says that for a given semi-stable vertex set, there exists a deformation retraction of C^{an} onto the subspace $\Gamma(C^{\text{an}}, \mathfrak{V})$.

Instead of a semi-stable vertex set we will often require the following less restrictive notion.

Definition 2.5. A *weak semi-stable vertex set* in C^{an} is defined to be any finite collection of points whose complement in C^{an} decomposes into the disjoint union of open subspaces of one of the three following types: a Berkovich open disk, an open annulus or a punctured Berkovich open disk.

Let \mathfrak{W} be the union of a finite number of k -points and a semi-stable vertex set. Then the space $C^{\text{an}} \setminus \mathfrak{W}$ decomposes into the disjoint union of open subspaces each isomorphic to one of the following: a Berkovich open disk, an open annulus or a punctured Berkovich open disk. Hence \mathfrak{W} is an example of a weak semi-stable vertex set.

Much like before, we define the skeleton $\Gamma(\mathfrak{W}, C^{\text{an}})$ associated to such a set. Let $\Gamma(\mathfrak{W}, C^{\text{an}})$ be the union of \mathfrak{W} and the skeleta of every open annulus and punctured open disk in the decomposition of $C^{\text{an}} \setminus \mathfrak{W}$. The closed subspace $\Gamma(\mathfrak{W}, C^{\text{an}})$ is homeomorphic to a connected, finite metric graph whose length function is not necessarily finite. Every edge which is the skeleton of a punctured Berkovich open disk can be identified with an interval of the form $(-\infty, a)$ for some $a \in \mathbb{R}$ which forces it to be of infinite length. As before, there exists a deformation retraction of C^{an} onto $\Gamma(\mathfrak{W}, C^{\text{an}})$.

2.4. The tangent space at a point on C^{an} . Once again let C be a smooth, projective, irreducible curve over the field k . We begin with the notion of the germ of a path. A path in C^{an} is a continuous function from a real interval J to C^{an} . If λ is a path and the interval has a designated initial point i , then the path is said to ‘start’ from the point $\lambda(i)$. In what follows we will need only finite real intervals, that is intervals of the form $[a, b]$, $(a, b]$, $[a, b)$ or (a, b) where $a < b$ are real numbers. We adopt the following convention concerning such intervals: a is the initial point and b the end point provided they lie in the interval.

Given an interval $[a, b]$ or $[a, b)$ we define an equivalence relation on the set of all paths from $[a, b)$ to C^{an} as follows. Let λ and λ' be two paths from $[a, b)$ to C^{an} . We set $\lambda \sim \lambda'$ if there exists $c \in (a, b)$ such that $\lambda|_{[a, c]} = \lambda'|_{[a, c]}$. The germ of a path λ is simply the equivalence class λ belongs to. We now define the tangent space at a point on C^{an} .

Definition 2.6. Let $x \in C^{\text{an}}$. The *tangent space at x* denoted T_x is the set of germs of paths from the interval $[0, 1]$ starting at x .

Let C' be another smooth, projective, irreducible curve over the field k and $\rho : C' \rightarrow C$ be a finite morphism. If $x' \in C'^{\text{an}}$ is any point then the tangent space at x' maps to the tangent space at $\rho(x')$ in an obvious fashion. Let $\lambda : [0, 1] \rightarrow C'^{\text{an}}$ be a representative of a point on the tangent space at x' . A germ of the path $\rho \circ \lambda$ defines an element of the tangent space $T_{\rho(x')}$. We have thus defined a map

$$\begin{aligned} d\rho_{x'} : T_{x'} &\rightarrow T_{\rho(x')} \\ \lambda &\mapsto \rho \circ \lambda \end{aligned}$$

2.4.1. *A description of how the tangent space varies for points of different type:* Given $x \in C^{\text{an}}$, we defined the non-archimedean valued complete field $\mathcal{H}(x)$ in Section 2.1. The set of elements $f \in \mathcal{H}(x)$ such that $|f(x)| \leq 1$ forms a local ring denoted $\mathcal{H}(x)^\circ$ whose maximal ideal $\mathcal{H}(x)^{\circ\circ}$ is defined by the formula $\{f \in \mathcal{H}(x) \mid |f(x)| < 1\}$. We refer to the quotient $\mathcal{H}(x)^\circ / \mathcal{H}(x)^{\circ\circ}$ as the residue field of $\mathcal{H}(x)$ and denote it $\widetilde{\mathcal{H}(x)}$.

If x is a point of type I or IV then there is exactly one point in the tangent space at x . If x is of type III then there are two distinct paths belonging to the tangent space at x . If x is of type II there are infinitely many points in the tangent space at x . The residue field $\widetilde{\mathcal{H}(x)}$ is of transcendence degree 1 over the residue field \tilde{k} and is hence the function field of a smooth, projective, irreducible curve over \tilde{k} . Let \tilde{C}_x denote this \tilde{k} -curve. We will subsequently write out a canonical bijection between the tangent space T_x and the closed points of this curve.

2.4.2. *The Non-Archimedean Poincaré-Lelong Theorem.* We state the non-archimedean Poincaré-Lelong theorem which will be used several times in Sections 3 and 4. What follows in this subsection is essentially a copy of what happens in a part of ([BPR]).

Let $x \in C^{\text{an}}$ be a point of type II. If $\text{Prin}(C)$ and $\text{Prin}(\tilde{C}_x)$ denote the group of principal divisors on the curves C/k and $\tilde{C}_{x/\tilde{k}}$ respectively then we define a map $\text{Prin}(C) \rightarrow \text{Prin}(\tilde{C}_x)$ as follows. Let $f \in k(C)$ be a rational function on C and c be any element in k such that $|f(x)| = |c|$. This implies that $(c^{-1} \cdot f) \in \mathcal{H}(x)^0$. Let f_x denote the image of $c^{-1} \cdot f$ in $\widetilde{\mathcal{H}(x)}$. Although $f_x \in \widetilde{\mathcal{H}(x)}$ depends on the choice of $c \in k$, the divisor f_x defines on \tilde{C}_x is independent of c . Hence we have a well defined map $\text{Prin}(C) \rightarrow \text{Prin}(\tilde{C}_x)$. It can be shown that this map is a homomorphism of groups.

A function $F : C^{\text{an}} \rightarrow \mathbb{R}$ is piecewise linear if for any path $\lambda : [a, b] \rightarrow C^{\text{an}}$ the composition $F \circ \lambda : [a, b] \rightarrow \mathbb{R}$ is piecewise linear. If x is a point of type II or III, $v \in T_x$ and $F : C^{\text{an}} \rightarrow \mathbb{R}$ is piecewise linear then we define the slope of the function F along v as follows. Let $\lambda : [0, 1] \rightarrow C^{\text{an}}$ be a representative of v . We set

$$\delta_v F(x) := \lim_{\epsilon \rightarrow 0} (F \circ \lambda)'(\epsilon).$$

Theorem 2.7. (Non-Archimedean Poincaré-Lelong Theorem) *Let $f \in k(C)$ be a non-zero rational function on the curve C and S denote the set of zeros and poles of f . Let \mathfrak{W} be a weak semi-stable vertex set whose set of k -points is the set S . Let $\Gamma(\mathfrak{W}, C^{\text{an}})$ be the skeleton associated to \mathfrak{W} and $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})} : [0, 1] \times C^{\text{an}} \rightarrow C^{\text{an}}$ be the deformation retraction with image $\Gamma(\mathfrak{W}, C^{\text{an}})$. We will use λ_e to denote the morphism $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}(1, -) : C^{\text{an}} \rightarrow C^{\text{an}}$. If $F := -\log|f| : C^{\text{an}} \setminus S \rightarrow \mathbb{R}$. Then we have that*

- (1) $F = F \circ \lambda_e$.
- (2) F is piecewise linear with integer slopes and F is linear on each edge of $\Gamma(\mathfrak{W}, C^{\text{an}})$.
- (3) If x is a type II point of C^{an} and v is an element of the tangent space T_x , then $\text{ord}_{\tilde{v}}(f_x) := \delta_v F(x)$ defines a discrete valuation $\text{ord}_{\tilde{v}}$ on the \tilde{k} -function field $\tilde{k}(\tilde{C}_x)$.
- (4) If $x \in C^{\text{an}}$ is of type II or III then $\sum_{v \in T_x} \delta_v F(x) = 0$.
- (5) Let $x \in S$, c be the ray in $\Gamma(\mathfrak{W}, C^{\text{an}})$ whose closure in C^{an} contains x and $y \in \mathfrak{W}$ the other end point of c . If $v \in T_y$ is that element of the tangent space T_y for which c is a representative, then $\delta_v F(y) = \text{ord}_x(f)$.

2.4.3. *An alternate description of the tangent space at a point x of type II.* Let $x \in C^{\text{an}}$ be a point of type II. We define the algebraic tangent space at a point of type II and show how this notion reconciles nicely with the definition we introduced

above. As x is of type II, the residue field $\widetilde{\mathcal{H}(x)}$ is of transcendence degree 1 over \tilde{k} . Uniquely associated to this \tilde{k} -function field is a smooth, projective \tilde{k} -curve. We denoted this curve \tilde{C}_x .

Definition 2.8. *The algebraic tangent space at x denoted T_x^{alg} is the set of closed points of the curve \tilde{C}_x .*

We now write out a map $B : T_x \rightarrow T_x^{\text{alg}}$. The closed points of the \tilde{k} -curve \tilde{C}_x correspond to multiplicative norms on the field $\widetilde{\mathcal{H}(x)}$. Given a germ $e_x \in T_x$ and $f \in \widetilde{\mathcal{H}(x)}$ there exists a $g \in \mathcal{H}(x)$ such that $|g(x)| = 1$ and $\tilde{g} = f$. Let $B(e_x)(f)$ be the slope of the function $-\log|g|$ along the germ e_x directed outwards. By the Non-Archimedean Poincare-Lelong Theorem the map B is a well defined bijection.

Let C' be a smooth, projective, irreducible curve over the field k and ρ a finite morphism $\rho : C' \rightarrow C$. If x' is a preimage of the point x then it must be of type II as well. The inclusion of non-archimedean valued complete fields $\mathcal{H}(x) \hookrightarrow \mathcal{H}(x')$ induces an extension of \tilde{k} -function fields $\widetilde{\mathcal{H}(x)} \hookrightarrow \widetilde{\mathcal{H}(x')}$. This defines a morphism $d\rho_{x'}^{\text{alg}} : T_{x'}^{\text{alg}} \rightarrow T_x^{\text{alg}}$ between the algebraic tangent space at x' and the algebraic tangent space at x . Recall that we have in addition a map $d\rho_{x'} : T_{x'} \rightarrow T_x$. These maps are compatible in the sense that the following diagram is commutative.

$$\begin{array}{ccc} T_{x'} & \xrightarrow{\quad d\rho_{x'} \quad} & T_x \\ \downarrow B & & \downarrow B \\ T_{x'}^{\text{alg}} & \xrightarrow{\quad d\rho_{x'}^{\text{alg}} \quad} & T_x^{\text{alg}} \end{array}$$

3. CONSTRUCTING A PAIR OF COMPATIBLE DEFORMATION RETRACTIONS

Let C and C' be smooth, projective curves over the field k and $\phi : C' \rightarrow C$ be a finite morphism between the curves such that the extension of function fields $k(C) \hookrightarrow k(C')$ is separable of degree n . In this section we will construct a compatible pair of deformation retractions

$$\psi : [0, 1] \times C^{\text{an}} \rightarrow C^{\text{an}}$$

and

$$\psi' : [0, 1] \times C'^{\text{an}} \rightarrow C'^{\text{an}}$$

whose images $\Upsilon_{C'^{\text{an}}} \subset C'^{\text{an}}$ and $\Upsilon_{C^{\text{an}}} \subset C^{\text{an}}$ are closed subspaces of C'^{an} and C^{an} respectively each with the structure of a connected, finite metric graph. Furthermore, the skeleta obtained are such that there exists weak semi-stable vertex sets $\mathfrak{A}' \subset C'^{\text{an}}$ and $\mathfrak{A} \subset C^{\text{an}}$ such that $\Upsilon_{C'^{\text{an}}} = \Gamma(\mathfrak{A}', C'^{\text{an}})$ and $\Upsilon_{C^{\text{an}}} = \Gamma(\mathfrak{A}, C^{\text{an}})$.

As discussed in the introduction ψ and ψ' will be **compatible** if the following diagram is commutative:

$$\begin{array}{ccc}
 [0, 1] \times C'^{\text{an}} & \xrightarrow{\psi'} & C'^{\text{an}} \\
 \downarrow id \times \phi^{\text{an}} & & \downarrow \phi^{\text{an}} \\
 [0, 1] \times C^{\text{an}} & \xrightarrow{\psi} & C^{\text{an}}
 \end{array}$$

When $C = \mathbb{P}_k^1$ and ϕ is a finite, separable morphism $C' \rightarrow \mathbb{P}_k^1$, Hrushovski and Loeser have constructed a compatible pair of deformation retractions for the spaces $\widehat{C'}$ and $\widehat{\mathbb{P}_k^1}$ onto Γ -internal subspaces in (Section 7, [HL]). A Γ -internal subset of \widehat{C} is similar to a finite, metric graph and $\widehat{C'}$ is much alike the Berkovich space C'^{an} . The construction of a compatible pair of deformation retractions which we will discuss in this section adapts the strategy outlined in (Section 7, [HL]).

Firstly, ϕ induces a finite, surjective morphism between the respective analytifications of the curve which can be described as follows. If $z \in C'^{\text{an}} \setminus C'(k)$ then z corresponds to a multiplicative semi-norm on the function field $k(C')$. The restriction of z to $k(C)$ defines the point $\phi^{\text{an}}(z)$. Alternately, if $z \in C^{\text{an}} \setminus C(k)$ then z corresponds to a multiplicative norm on the function field $k(C)$. The set of multiplicative norms on the function field $k(C')$ which extend the norm z on $k(C)$ is precisely the set of preimages of the point z , that is $\{(\phi^{\text{an}})^{-1}(z)\}$.

Let \mathfrak{R} be the finite set of k -points of C over which the morphism ϕ is ramified and \mathfrak{V} be a semi-stable vertex set. If $\mathfrak{W} := \mathfrak{V} \cup \mathfrak{R}$ then \mathfrak{W} is a weak semi-stable vertex set and as discussed in 2.3, there exists a connected, finite metric graph $\Gamma(\mathfrak{W}, C^{\text{an}})$ with a deformation retraction

$$\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})} : [0, 1] \times C^{\text{an}} \rightarrow C^{\text{an}}$$

whose image is $\Gamma(\mathfrak{W}, C^{\text{an}})$.

Let $p \in C^{\text{an}}$. There exists a path $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^p : [0, 1] \rightarrow C^{\text{an}}$ starting from p and ending at a point belonging to $\Gamma(\mathfrak{W}, C^{\text{an}})$, namely the path given by $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^p(t) := \lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}(t, p)$ for $t \in [0, 1]$.

Let $y \in C(k)$ be a closed point $\phi^{-1}(y) = \{y'_1, \dots, y'_n\}$ be the set of preimages of y . We ask the following question motivated by our desire to construct a deformation retraction which ‘lifts’ a part of $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y$. *Can one lift a portion of the path starting from y to a set of paths starting from each of the y_j and if this is so, then what would be the maximum possible portion?* Since the points over which the morphism ϕ is ramified belongs to the set \mathfrak{W} we restrict our attention to only those k -points over which the morphism is unramified. Though it is intuitively clear what we mean by the lift of a path, we provide a definition.

Definition 3.1. Let $a < b$ be real numbers and $u : [a, b] \rightarrow C^{\text{an}}$ be a continuous function. A *lift of the path u* is a path $u' : [a, b] \rightarrow C'^{\text{an}}$ such that $u = \phi \circ u'$.

By (Lemma 7.4.1, [HL]), for any j there exists open neighbourhoods N_{y_j} and N_y around y_j and y respectively such that the morphism ϕ restricted to N_{y_j} is a homeomorphism of N_{y_j} onto N_y . Consequently, there exists a $t \in (0, 1]$ such that $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y|_{[0, t]}$ lifts uniquely to a path in C'^{an} starting from y_j and terminating at some pre-image of $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y(t)$. To answer the second part of the question

we introduce the notion of a forward branching point associated to a deformation retraction motivated by Definition 7.5.2 in [HL].

3.1. Forward Branching points. We continue our discussion in the context of the curves C and C' with the morphism $\phi : C' \rightarrow C$. Let \mathfrak{B} be a weak semi-stable vertex set for the space C^{an} . As discussed in 2.3, associated to this finite set is the skeleton $\Gamma(\mathfrak{B}, C^{\text{an}})$ and a deformation retraction $\lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})}$ of the space C^{an} onto $\Gamma(\mathfrak{B}, C^{\text{an}})$. We define a forward branching point for the retraction $\lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})}$ and the morphism ϕ as follows.

Definition 3.2. Let $q \in C'^{\text{an}}$ and $z \in C^{\text{an}}$ be a point of type I or IV, such that $\phi^{\text{an}}(q) \in \lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})}^z([0, 1])$ and $\phi^{\text{an}}(q) \notin \Gamma(\mathfrak{B}, C^{\text{an}})$. Let $r \in [0, 1]$ be that real number for which $\phi(q) = \lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})}^z(r)$. Then $q \in C'^{\text{an}}$ is a *forward branching point* for the pair $(\phi, \lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})})$ if for every $\epsilon \in (r, 1]$, there exists atleast two distinct lifts of the path $\lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})}^z|_{[r, \epsilon]}$ starting from q .

The definition above is independent of our choice of the point z . This follows from the fact that $\lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})}$ is well defined and if z_1 and z_2 are points of either type I or IV such that there exists real numbers r_{z_1} and r_{z_2} belonging to $[0, 1]$ for which $\lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})}^{z_1}(r_{z_1}) = \lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})}^{z_2}(r_{z_2})$ then $\lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})}^{z_1}|_{[r_{z_1}, 1]} = \lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})}^{z_2}|_{[r_{z_2}, 1]}$. Furthermore, the r_{z_i} can be chosen so that they are minimal for this condition. We now show that the number of forward branching points is finite.

Proposition 3.3. *The number of forward branching points for the pair $(\phi, \lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})})$ in C'^{an} is finite.*

Proof. We first show that it suffices to prove the proposition when $C = \mathbb{P}_k^1$ and $\mathfrak{B} = \{\eta_{0,1}\}$. In this case the image of the deformation retraction $\lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})}$ will be the single point $\eta_{0,1}$.

Lemma 3.4. *Suppose that the smooth projective curve C' defined over k is such that for every finite morphism $\alpha : C' \rightarrow \mathbb{P}_k^1$ the number of forward branching points for the pair $(\alpha, \lambda_{\Gamma(\eta_{0,1}, \mathbb{P}_k^{1,\text{an}})})$ in C'^{an} is finite. Then the number of forward branching points for the pair $(\phi, \lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})})$ in C'^{an} is finite.*

Proof. Let $\beta : C \rightarrow \mathbb{P}_k^1$ be a finite morphism. We apply the hypotheses to the morphism $\alpha := \beta \circ \phi$. We claim that barring a finite number, every forward branching point for $(\phi, \lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})})$ in C'^{an} is a forward branching point for $(\alpha, \lambda_{\Gamma(\eta_{0,1}, \mathbb{P}_k^{1,\text{an}})})$ in C'^{an} .

Let $q' \in C'^{\text{an}}$ be such that $\alpha^{\text{an}}(q') \neq \eta_{0,1}$ and q' is a forward branching point for $(\phi, \lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})})$. Let $q := \phi^{\text{an}}(q')$. For suitably small r , the number of lifts of the path $\lambda_{\Gamma(\eta_{0,1}, \mathbb{P}_k^{1,\text{an}})}(-, \alpha^{\text{an}}(q')) : [0, r] \rightarrow \mathbb{P}_k^{1,\text{an}}$ which start at q' for the morphism α^{an} is equal to the product of the number of lifts of the path $\lambda_{\Gamma(\eta_{0,1}, \mathbb{P}_k^{1,\text{an}})}(-, \phi^{\text{an}}(q')) : [0, r] \rightarrow \mathbb{P}_k^{1,\text{an}}$ which start at q for the morphism β^{an} and the number of lifts of the path $\lambda_{\Gamma(\mathfrak{B}, C^{\text{an}})}(-, q) : [0, r] \rightarrow C^{\text{an}}$ which start at q' for the morphism ϕ^{an} . The claim made above follows from this observation and hence the proof of the lemma. \square

We now prove the proposition for a pair $(\alpha, \lambda_{\Gamma(\eta_{0,1}, \mathbb{P}_k^{1,\text{an}})})$ where $\alpha : C' \rightarrow \mathbb{P}_k^1$ is a finite morphism. By Section 2.2, outside the point ∞ every point of $\mathbb{P}_k^{1,\text{an}}$ which is not of type IV corresponds uniquely to a closed ball contained in k or an element of the field k . For the time being we allow for closed balls of radius 0. Let b be a closed ball and $n(\alpha, b)$ be the number of preimages of this closed ball for the morphism α^{an} . Observe that if $q \in C'^{\text{an}}$ is a forward branching point and $b := \alpha^{\text{an}}(q)$ of radius r then there exists a ball b' of radius $r' > r$ such that for every ball $b \subset b^* \subset b'$

we have that $n(\alpha, b) < n(\alpha, b^*)$. The proposition now follows from Lemma 7.5.4 in [HL]. \square

3.2. The Construction of a compatible pair of deformation retractions.

We now try and answer the second part of the question we had posed at the beginning of Section 3. Recall that we used \mathfrak{R} to denote the finite set of k -points of C over which the morphism ϕ is ramified. Let \mathfrak{V} be a semi-stable vertex set. We defined $\mathfrak{W} := \mathfrak{V} \cup \mathfrak{R}$.

Proposition 3.5. *Let $y \in C(k)$ be a closed point and y' be one of its pre-images for the morphism ϕ . Let t_y denote the smallest real number in the interval $[0, 1]$ such that there exists a forward branching point $p' \in C'^{\text{an}}$ for which $\phi(p') = \lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y(t_y)$. Then $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y|_{[0, t_y]}$ admits a unique lift starting at y' .*

Proof. We divide the proof of the proposition into two parts. The weak semi-stable vertex set \mathfrak{W} contains the finite set of k -points of C over which the morphism ϕ is ramified. Hence we need only prove the proposition when $y \in C(k)$ is a closed point over which the morphism ϕ is unramified. We begin by showing that the path $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y|_{[0, t_y]}$ admits atleast one lift starting at y' .

Existence : The collection of points $a \in [0, 1]$ for which there exists a lift of the path $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y|_{[0, a]}$ is a closed subspace of $[0, 1]$. It follows that this collection admits a supremum. Let $a_{y'}$ denote the largest real number belonging to the interval $[0, t_y]$ such that there exists a lift of the path $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y|_{[0, a_{y'}]}$ starting from y' .

Since ϕ is etale over the point y , there exists a portion of the path $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y|_{[0, t_y]}$ which can be lifted. This implies that $a_{y'}$ must be greater than zero. Arguing by contradiction, we will show that $a_{y'} = t_y$. Let $a_{y'}$ be strictly less than t_y and $q' \in C^{\text{an}}$ be a pre-image of $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y(a_{y'})$ such that there exists a path starting from y' and ending at q' which is a lift of the path $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y|_{[0, a_{y'}]}$. The points $q' \in C'^{\text{an}}$ and $\phi(q') \in C^{\text{an}}$ correspond to multiplicative norms on the function fields $k(C')$ and $k(C)$ respectively. This induces an inclusion of the function fields $\mathcal{H}(\widetilde{\phi(q')}) \hookrightarrow \mathcal{H}(\widetilde{q'})$. It follows that the morphism ϕ induces a surjective morphism between the algebraic tangent spaces (cf. Definition 2.8) at q' and $\phi(q')$. Since there is a canonical bijection between the geometric tangent space (cf. Definition 2.6) and the algebraic tangent space at a type II point, there exists a lift of the path $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y|_{[a_{y'}, a_{y'} + \epsilon]}$ for suitably small ϵ . The fact that q' is not a forward branching point for $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y$ implies that this lift is unique. Gluing this lift with the lift of the path $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y|_{[0, a_{y'}]}$ implies a contradiction.

Uniqueness : We now show that there can be at most one lift of the path $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y|_{[0, t_y]}$.

Let λ'_1 and λ'_2 be two lifts of the path $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y|_{[0, t_y]}$. Let a' be the largest real number in the interval $[0, t_y]$ such that $\lambda'_1|_{[0, a']} = \lambda'_2|_{[0, a']}$. We need to show that a' is equal to t_y . Assume for arguments sake that this is not true. By our choice of a' we have that $\lambda'_1(a') = \lambda'_2(a')$. From our definition of the real number t_y , there exists a *unique* lift of the path $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y|_{[a', a' + \epsilon]}$ for suitably small ϵ starting from $\lambda'_i(a')$. This implies that $\lambda'_1|_{(0, a' + \epsilon)} = \lambda'_2|_{(0, a' + \epsilon)}$ which gives us our contradiction. \square

Observe that every point $p \in C^{\text{an}}$ which is of type II or III lies on a path of the form $\lambda_{\Gamma(\mathfrak{W}, C^{\text{an}})}^y|_{[0, 1]}$ where y is an unramified closed point. As a corollary to the above proposition we have:

Corollary 3.6. *Let $p \in C^{\text{an}}$ be a point which is not of type IV and p' be one of its pre-images for the morphism ϕ . Let t_p denote the smallest real number in the*

interval $[0, 1]$ such that there exists a forward branching point $p' \in C'^{\text{an}}$ for which $\phi^{\text{an}}(p') = \lambda_{\Gamma(\mathfrak{V}, C^{\text{an}})}^p(t_p)$. Then $\lambda_{\Gamma(\mathfrak{V}, C^{\text{an}})}^p|_{[0, t_p]}$ admits a unique lift starting at p' .

We are now in a position to prove the existence of a pair of compatible deformation retractions. Recall that \mathfrak{V} is a semi-stable vertex set for C^{an} . Let $F \subset C^{\text{an}}$ denote the union of the set of images of the forward branching points for the deformation retraction $\lambda_{\Gamma(\mathfrak{V}, C^{\text{an}})}$ and the set of k -points over which the morphism ϕ is ramified. Let \mathfrak{V}_0 be a weak semi-stable vertex set containing $\mathfrak{V} \cup F$. Note that $\Gamma(\mathfrak{V}, C^{\text{an}}) \subset \Gamma(\mathfrak{V}_0, C^{\text{an}})$ and

$$(\lambda_{\Gamma(\mathfrak{V}_0, C^{\text{an}})})|_{C^{\text{an}} \setminus \Gamma(\mathfrak{V}_0, C^{\text{an}})} = (\lambda_{\Gamma(\mathfrak{V}, C^{\text{an}})})|_{C^{\text{an}} \setminus \Gamma(\mathfrak{V}_0, C^{\text{an}})}.$$

It follows that the deformation retraction $\lambda_{\Gamma(\mathfrak{V}_0, C^{\text{an}})}$ does not have any forward branching points. Extending our discussion above, we can define a deformation retraction of C'^{an} onto $\phi^{-1}(\Gamma(\mathfrak{V}_0, C^{\text{an}}))$ which is compatible with $\lambda_{\Gamma(\mathfrak{V}_0, C^{\text{an}})}$. However, our stated goal was to construct a pair of compatible deformation retractions whose images are *connected*, finite graphs which in addition are of the form $\Gamma(\mathfrak{A}', C'^{\text{an}})$ and $\Gamma(\mathfrak{A}, C^{\text{an}})$ where \mathfrak{A} and \mathfrak{A}' are weak semi-stable vertex sets contained in C^{an} and C'^{an} respectively.

This can be accomplished by enlarging the finite graph $\Gamma(\mathfrak{V}_0, C^{\text{an}})$ suitably so that its pre-image for the morphism ϕ becomes path connected.

Proposition 3.7. *Let $\phi : C' \rightarrow C$ be a finite morphism such that the extension of function fields $k(C) \hookrightarrow k(C')$ is separable. There exists a pair of compatible definable retractions $\psi : [0, 1] \times C^{\text{an}} \rightarrow C^{\text{an}}$ and $\psi' : [0, 1] \times C'^{\text{an}} \rightarrow C'^{\text{an}}$ such that their images are connected, finite graphs associated to weak semi-stable vertex sets.*

Proof. We proceed as outlined above by first constructing the compatible pair of deformation retractions $\lambda_{\Gamma(\mathfrak{V}_0, C^{\text{an}})}$ and λ' whose images are $\Gamma(\mathfrak{V}_0, C^{\text{an}})$ and $\Gamma'(C^{\text{an}}) := \phi^{-1}(\Gamma(\mathfrak{V}_0, C^{\text{an}}))$. We then enlarge $\Gamma(\mathfrak{V}_0, C^{\text{an}})$ suitably so that it remains a finite graph and the pre-image of the enlarged graph for the morphism ϕ is pathwise connected. We call this larger graph $\Upsilon_{C^{\text{an}}}$. There then exists a compatible pair of deformation retractions $\psi : [0, 1] \times C^{\text{an}} \rightarrow C^{\text{an}}$ and $\psi' : [0, 1] \times C'^{\text{an}} \rightarrow C'^{\text{an}}$ whose images are the connected, finite graphs $\Upsilon_{C^{\text{an}}}$ and $\phi^{-1}(\Upsilon_{C^{\text{an}}})$ respectively.

Let $q' \in C'^{\text{an}}$. Let $t_{\phi(q')} \in [0, 1]$ be such that $\lambda_{\Gamma(\mathfrak{V}_0, C^{\text{an}})}(a, \phi(q')) \cap \Gamma(\mathfrak{V}_0, C^{\text{an}}) = \emptyset$ for every $0 \leq a < t_{\phi(q')}$ and $\lambda_{\Gamma(\mathfrak{V}_0, C^{\text{an}})}(t_{\phi(q')}, \phi(q')) \in \Gamma(\mathfrak{V}_0, C^{\text{an}})$. By Corollary 3.4, there exists a unique lift of the path $\lambda_{\Gamma(\mathfrak{V}_0, C^{\text{an}})}(_, \phi(q'))|_{[0, t_{\phi(q')}]}$ starting at the point q' . Let this unique lift be called $\lambda'^{q'}$. More precisely, $\lambda'^{q'}$ is the unique continuous function from $[0, t_{\phi(q')}]$ to C'^{an} such that $\phi \circ \lambda'^{q'} = \lambda_{\Gamma(\mathfrak{V}_0, C^{\text{an}})}(_, \phi(q'))|_{[0, t_{\phi(q')}]}$. We define $\lambda' : [0, 1] \times C'^{\text{an}} \rightarrow C'^{\text{an}}$ as follows. For $0 \leq a \leq t_{\phi(q')}$ we set $\lambda'(a, q') := \lambda'^{q'}(a)$ and for $a \in (t_{\phi(q')}, 1]$ let $\lambda'(a, q') := \lambda'^q(t_{\phi(q')})$. The map λ' is a deformation retraction of C'^{an} and is compatible with the deformation retraction $\lambda_{\Gamma(\mathfrak{V}_0, C^{\text{an}})}$. Observe that since the morphism ϕ is finite, $\phi^{-1}(\Gamma(\mathfrak{V}_0, C^{\text{an}}))$ is also homeomorphic to a finite graph.

Let \mathfrak{A}' be a weak-semi stable vertex set contained in C'^{an} with the following properties.

- (1) $\mathfrak{A}' = (\phi^{\text{an}})^{-1}(\phi^{\text{an}}(\mathfrak{A}'))$.
- (2) The finite graph $\Gamma(\mathfrak{A}', C'^{\text{an}})$ contains $\phi^{-1}(\Gamma(\mathfrak{V}_0, C^{\text{an}}))$ and is pathwise connected.

The image $\phi^{\text{an}}(\Gamma(\mathfrak{A}', C'^{\text{an}}))$ contains $\Gamma(\mathfrak{V}_0, C^{\text{an}})$ and hence there exists a weak semi-stable vertex set $\mathfrak{A} \subset C^{\text{an}}$ containing \mathfrak{V}_0 such that

$$\phi^{\text{an}}(\Gamma(\mathfrak{A}', C'^{\text{an}})) = \Gamma(\mathfrak{A}, C^{\text{an}})$$

and $\Gamma(\mathfrak{A}, C^{\text{an}})$ is pathwise connected.

Let $\Upsilon_{C^{\text{an}}} := \Gamma(\mathfrak{A}, C^{\text{an}})$. There exists a canonical deformation retraction $\psi : [0, 1] \times C^{\text{an}} \rightarrow C^{\text{an}}$ with image $\Upsilon_{C^{\text{an}}}$. One need only imitate the construction of the deformation retraction λ' given $\lambda_{\Gamma(\mathfrak{A}_0, C^{\text{an}})}$ to obtain $\psi' : [0, 1] \times C'^{\text{an}} \rightarrow C'^{\text{an}}$ such that the pair ψ and ψ' are compatible. Our choice of \mathfrak{A}' will imply that $\Gamma(\mathfrak{A}', C'^{\text{an}}) = (\phi^{\text{an}})^{-1}(\Upsilon_{C^{\text{an}}})$. We set $\Upsilon_{C'^{\text{an}}} := \Gamma(\mathfrak{A}', C'^{\text{an}})$. Hence the proof. \square

We assumed that the morphism $\phi : C' \rightarrow C$ is such that the extension of function fields $k(C) \hookrightarrow k(C')$ is only separable. For the following proposition we assume that the extension is in addition Galois. Let $G := \text{Gal}(k(C')/k(C))$. We will require the following proposition in Section 4.

Proposition 3.8. *If $\sigma \in G$ then*

$$\psi' \circ (id_{[0,1]} \times \sigma) = \sigma \circ \psi'.$$

Proof. Every element of the Galois group induces a homeomorphism of C'^{an} onto itself. Let $p' \in C'^{\text{an}}$ be a point which is not of type IV and $p := \phi^{\text{an}}(p')$. The paths $\psi'(-, \sigma(p'))$ and $\sigma \circ \psi'(-, p')$ are lifts of the path $\psi(-, p)$ starting from $\sigma(p')$. By Corollary 3.5, $\psi'(-, \sigma(p')) = \sigma \circ \psi'(-, p')$. Note that it is sufficient to have verified the statement for points of type I, II and III and we conclude our proof. \square

4. CALCULATING THE GENUS OF THE GRAPH $\Upsilon_{C'^{\text{an}}}$

We begin by explicitly describing a vertex set and a set of edges for the skeleta $\Upsilon_{C^{\text{an}}}$ and $\Upsilon_{C'^{\text{an}}}$. We will denote the sets of vertices $V(\Upsilon_{C^{\text{an}}})$ and $V(\Upsilon_{C'^{\text{an}}})$ and the sets of edges $E(\Upsilon_{C^{\text{an}}})$ and $E(\Upsilon_{C'^{\text{an}}})$ respectively. We will require that the two vertex sets satisfy the following conditions.

- (1) $V(\Upsilon_{C'^{\text{an}}}) = (\phi^{\text{an}})^{-1}(V(\Upsilon_{C^{\text{an}}}))$.
- (2) From the construction in the previous section, there exists weak semi-stable vertex sets $\mathfrak{A} \subset C^{\text{an}}$ and $\mathfrak{A}' \subset C'^{\text{an}}$ such that $\Upsilon_{C^{\text{an}}} = \Gamma(\mathfrak{A}, C^{\text{an}})$ and $\Upsilon_{C'^{\text{an}}} = \Gamma(\mathfrak{A}', C'^{\text{an}})$. We will require that $\mathfrak{A} \subset V(\Upsilon_{C^{\text{an}}})$ and $\mathfrak{A}' \subset V(\Upsilon_{C'^{\text{an}}})$.
- (3) If p (p') is a point on the skeleta $\Upsilon_{C^{\text{an}}}$ ($\Upsilon_{C'^{\text{an}}}$) for which there exists a sufficiently small open neighbourhood $U \subset \Upsilon_{C^{\text{an}}}$ ($U' \subset \Upsilon_{C'^{\text{an}}}$) such that $U \setminus \{p\}$ ($U' \setminus \{p'\}$) has at least three connected components then $p \in V(\Upsilon_{C^{\text{an}}})$ ($p' \in V(\Upsilon_{C'^{\text{an}}})$).

It can be shown easily that a pair $(V(\Upsilon_{C^{\text{an}}}), V(\Upsilon_{C'^{\text{an}}}))$ satisfying the properties described does indeed exist. Having defined the vertex sets $V(\Upsilon_{C^{\text{an}}})$ and $V(\Upsilon_{C'^{\text{an}}})$ the set of edges for the skeleton $\Upsilon_{C^{\text{an}}}$ ($\Upsilon_{C'^{\text{an}}}$) will be the collection of all paths contained in $\Upsilon_{C^{\text{an}}}$ ($\Upsilon_{C'^{\text{an}}}$) connecting any two vertices. Since $\Upsilon_{C^{\text{an}}}$ ($\Upsilon_{C'^{\text{an}}}$) is the skeleton associated to a weak semi-stable vertex set, the edges of the skeleton are identified with real intervals. This defines a length function on the set of edges.

The genus of the skeleton $\Upsilon_{C^{\text{an}}}$, ($\Upsilon_{C'^{\text{an}}}$) is given by the formula $1 - V(\Upsilon_{C^{\text{an}}}) + E(\Upsilon_{C^{\text{an}}})$, ($1 - V(\Upsilon_{C'^{\text{an}}}) + E(\Upsilon_{C'^{\text{an}}})$). It is independent of the choice of the respective vertex set and the set of edges. We use $g(\Upsilon_{C^{\text{an}}})$, ($g(\Upsilon_{C'^{\text{an}}})$) to denote the genus of $\Upsilon_{C^{\text{an}}}$, ($\Upsilon_{C'^{\text{an}}}$).

For any point of type II, let g_p denote the genus of the residue field $\widehat{\mathcal{H}(p)}$ and if p is not of type II then we set $g_p = 0$. The genus formula (4.5, [AB]) implies that

$$(4) \quad g(C) = g(\Upsilon_{C^{\text{an}}}) + \sum_{p \in V(\Upsilon_{C^{\text{an}}})} g_p.$$

Employing the same reasoning as above we obtain a corresponding formula for the curve C' .

$$(5) \quad g(C') = g(\Upsilon_{C'^{\text{an}}}) + \sum_{p' \in V(\Upsilon_{C'^{\text{an}}})} g_{p'}.$$

The morphism $\phi : C' \rightarrow C$ is a finite separable morphism between smooth, projective curves. The Riemann-Hurwitz formula (Corollary IV.2.4, [H]) enables us to relate the genera of the curves C' and C . Precisely,

$$(6) \quad 2g(C') - 2 = \deg(\phi)(2g(C) - 2) + R$$

where R is a divisor on the curve C' such that if ϕ is tamely ramified at $x' \in C'$ then $\text{ord}_{x'}(R) = \text{ram}(x', x) - 1$. Using equations (4), (5) and (6) above, we obtain the following formula relating the genera of the skeleta $g(\Upsilon_{C^{\text{an}}})$ and $g(\Upsilon_{C'^{\text{an}}})$.

$$2g(\Upsilon_{C'^{\text{an}}}) - 2 + 2(\sum_{p' \in V(\Upsilon_{C'^{\text{an}}})} g_{p'}) = \deg(\phi)(2g(\Upsilon_{C^{\text{an}}}) - 2) + \deg(\phi)(2\sum_{p \in V(\Upsilon_{C^{\text{an}}})} g_p) + R$$

In the above equation the only vertices in $V(\Upsilon_{C^{\text{an}}})$ and $V(\Upsilon_{C'^{\text{an}}})$ for which the genera g_p or $g_{p'}$ are not necessarily zero are those of type II. Hence we may restrict our attention to those vertices of type II. If $p' \in V(\Upsilon_{C'^{\text{an}}})$ then $\phi^{\text{an}}(p') = p$. The point $p \in C^{\text{an}}$ corresponds to a multiplicative norm on the function field $k(C)$ and the collection of norms on $k(C')$ which extend p are precisely the set of preimages $(\phi^{\text{an}})^{-1}(p)$. The extension of non-archimedean valued complete fields $\mathcal{H}(p) \hookrightarrow \mathcal{H}(p')$ induces an extension of \tilde{k} -function fields $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$. The \tilde{k} -function fields $\widetilde{\mathcal{H}(p)}$ and $\widetilde{\mathcal{H}(p')}$ both correspond to smooth projective \tilde{k} curves which we denote by \tilde{C}_p and $\tilde{C}_{p'}$ respectively. If the morphism $\tilde{C}_{p'} \rightarrow \tilde{C}_p$ was separable then the Riemann-Hurwitz formula relates the numbers $g_{p'}$ and g_p .

Let $I(p', p)$ denote the subfield of $\widetilde{\mathcal{H}(p')}$ which contains $\widetilde{\mathcal{H}(p)}$ such that $I(p', p)$ is purely inseparable over $\widetilde{\mathcal{H}(p)}$ of degree $i(p)$ and $I(p', p) \hookrightarrow \widetilde{\mathcal{H}(p)}$ is a separable field extension of degree $i(p')$. The function field $I(p', p)$ corresponds to a smooth projective \tilde{k} -curve which is homeomorphic to \tilde{C}_p . We denote this curve $W(p', p)$. Since $I(p', p)$ is purely inseparable over the function field $\widetilde{\mathcal{H}(p)}$ the genus of the curve $W(p', p)$ is equal to g_p . Applying the Riemann-Hurwitz formula to the extension $I(p', p) \hookrightarrow \widetilde{\mathcal{H}(p')}$ and substituting in the equation above gives

$$2g(\Upsilon_{C'^{\text{an}}}) - 2 = \deg(\phi) \cdot (2g(\Upsilon_{C^{\text{an}}}) - 2) + \sum_{p \in V(\Upsilon_{C^{\text{an}}})} A_p(2g_p - 2) + \sum_{p \in V(\Upsilon_{C^{\text{an}}})} 2(n_p + \deg(\phi)) + R + \sum_{p' \in V(\Upsilon_{C'^{\text{an}}})} R_{p'}.$$

The constant A_p is defined to be $\sum_{p' \in (\phi^{\text{an}})^{-1}(p)} i(p') + \deg(\phi)$. The number of preimages of a point $p \in C^{\text{an}}$ for the morphism ϕ^{an} is denoted n_p . The ramification divisor associated to the Riemann-Hurwitz formula for the field extension $I(p', p) \hookrightarrow \widetilde{\mathcal{H}(p')}$ is denoted $R_{p'}$.

In what follows in the rest of this section we will study the constants arising in this formula. Notably the numbers $i(p')$ and the *local* ramification divisors $R_{p'}$. In the next section we study the variable n_p under the additional hypothesis that the extension of function fields $k(C) \hookrightarrow k(C')$ is Galois.

4.1. Calculating $i(p')$ and the defect. Let M be a non-archimedean valued field with valuation v . Let vM denote the value group and Mv denote the residue field. Let M' be a finite extension of the field M such that the valuation v extends uniquely to M' . By Ostrowski's lemma we have the following equality.

$$[M' : M] = (vM' : vM)[M'v : Mv]p^r$$

Here p is the characteristic of the residue field if it is positive and one otherwise. The value $d(M', M) := p^r$ is called the *defect* of the extension. If $r = 0$ then we call the extension M'/M *defectless*.

We now relate this definition to the situation we are dealing with. Let p be a point of type II belonging to C^{an} and $p' \in (\phi^{\text{an}})^{-1}(p)$. Since the field k is

algebraically closed non-archimedean valued and the points p and p' are of type II, the value group of the fields $\mathcal{H}(p)$ and $\mathcal{H}(p')$ remain the same. We hence have the following equality

$$[\mathcal{H}(p') : \mathcal{H}(p)] = [\widetilde{\mathcal{H}(p')} : \widetilde{\mathcal{H}(p)}].d(p', p)$$

We now show the following.

Lemma 4.1. *Let $p \in C^{\text{an}}$ and $p' \in (\phi^{\text{an}})^{-1}(p)$. The extension $\mathcal{H}(p) \hookrightarrow \mathcal{H}(p')$ is defectless. That is the defect $d(p', p) = 1$.*

Proof. We will make use of the Poincaré-Lelong theorem and our construction in Section 3 of the pair of compatible deformation retractions ψ and ψ' . Let $r \in [0, 1]$ be the smallest real number such that $p \in \psi(r, C^{\text{an}})$. Since the deformation retractions are compatible it follows that if $p' \in (\phi^{\text{an}})^{-1}(p)$ then $p' \in \psi'(r, C'^{\text{an}})$.

Let $x \in C(k)$ be such that $\psi(r, x) = p$. Let e_x denote the path $\psi(_, x) : [0, r] \rightarrow C^{\text{an}}$. Hence e_x belongs to the tangent space T_p at the point p . Let t_x be a uniformisant of the local ring $O_{C, x}$ such that $t_x(p) = 1$ and \tilde{t}_x - the image of t_x in the residue field $\widetilde{\mathcal{H}(p)}$ is a uniformisant at the point e_x in $\widetilde{\mathcal{H}(p)}$. In addition we choose t_x so that it has no zeros or poles at any k -point y for which the path $\psi(_, y) : [0, t] \rightarrow C^{\text{an}}$ coincides with e_x in the tangent space. Such a choice is possible by the semi-stable decomposition associated to the skeleton $\Upsilon_{C^{\text{an}}}$. Let t'_x denote the image of the t_x in the function field $k(C')$. Our choice of t_x implies that for every $p' \in (\phi^{\text{an}})^{-1}(p)$, $|t'_x(p')| = 1$.

The inclusion $\mathcal{H}(p) \hookrightarrow \mathcal{H}(p')$ induces an inclusion of \tilde{k} -function fields $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$. We call \tilde{C}_p and $\tilde{C}'_{p'}$ the smooth projective curves associated to these function fields. The path e_x introduced above is a \tilde{k} -point of the curve \tilde{C}_p . Let $E(x, p')$ denote the set of preimages of this point on the curve $\tilde{C}'_{p'}$.

Let $S := \{x'_1, \dots, x'_k\}$ denote the preimages of the point x and $\text{ram}(x'_i, x)$ denote the ramification index of the morphism ϕ at the point x'_i . Since the skeleton $\Upsilon_{C^{\text{an}}}$ contains the set of k -points over which the morphism is ramified, we have that $\text{ram}(x'_i, x) = 1$ for all i .

The set of paths $\{\psi'(_, x'_i) : [0, r] \rightarrow C'^{\text{an}}\}$ coincides with the set $E(e_x) := \cup_{p' \in (\phi^{\text{an}})^{-1}(p)} E(e_x, p')$. Our choice of t_x implies that t'_x cannot have a zero or pole at any point $y' \in C'(k)$ for which $\psi'(_, y') : [0, r] \rightarrow C'^{\text{an}} \in E(e_x)$ and $y' \notin \phi^{-1}(x)$.

Let $p' \in (\phi^{\text{an}})^{-1}(p)$ and $e' \in E(e_x, p')$. Let $S_{e', p'}$ be the collection of those $x' \in S$ such that $\psi'(r, x') = p'$ and $\psi'(_, x') : [0, r] \rightarrow C'^{\text{an}} = e'$. The non-archimedean Poincaré-Lelong theorem implies that

$$\delta_{e'}(-\log(t'_x))(p') = \sum_{x' \in S_{e', p'}} \text{ram}(x', x) = \#(S_{e', p'}).$$

The second equality follows from the fact that $\text{ram}(x', x) = 1$. Furthermore,

$$\delta_{e'}(-\log(t'_x))(p') = \text{ord}_{e'}(\tilde{t}'_x).$$

Since $\sum_{e' \in E(e_x, p')} \text{ord}_{e'}(\tilde{t}'_x) = [\widetilde{\mathcal{H}(p')} : \widetilde{\mathcal{H}(p)}]$, it follows that

$$\sum_{p' \in (\phi^{\text{an}})^{-1}(p)} [\widetilde{\mathcal{H}(p')} : \widetilde{\mathcal{H}(p)}] = \sum_{x' \in S} \text{ram}(x', x) = \sum_{e', p'} \#(S_{e', p'}).$$

Hence we have that

$$\sum_{p' \in (\phi^{\text{an}})^{-1}(p)} [\widetilde{\mathcal{H}(p')} : \widetilde{\mathcal{H}(p)}] = \#(S).$$

As the field k is algebraically closed, the expression on the right is equal to the degree of the morphism ϕ and we have that

$$\sum_{p' \in (\phi^{\text{an}})^{-1}(p)} [\widetilde{\mathcal{H}(p')} : \widetilde{\mathcal{H}(p)}] = \sum_{p' \in (\phi^{\text{an}})^{-1}(p)} [\mathcal{H}(p') : \mathcal{H}(p)].$$

Thus showing that for every $p' \in (\phi^{\text{an}})^{-1}(p)$, the extension $\mathcal{H}(p) \hookrightarrow \mathcal{H}(p')$ is defectless. \square

Let $p \in C^{\text{an}}$ be a point of type II and $p' \in (\phi^{\text{an}})^{-1}(p)$. We now study the separable and inseparable degree of the extension of \tilde{k} -function fields $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$. Let $r \in [0, 1]$ be such that $p \in \psi(r, C^{\text{an}})$. Let \sim_j be an equivalence relation on the set $Q_p := \{x' \in C'(k) \mid \psi'(r, x') = p'\}$ defined as follows. We set $x'_1 \sim_j x'_2$ if and only if $\phi(x'_1) = \phi(x'_2)$ and the paths $\psi'(-, x') : [0, r] \rightarrow C'^{\text{an}}$ and $\psi'(-, x') : [0, r] \rightarrow C'^{\text{an}}$ coincide as elements of the tangent space $T_{p'}$ at p' . We denote the equivalence class of an element $x' \in R_p$ by $[x']_j$ and its cardinality $\sharp[x']_j$.

Proposition 4.2. *Let $j(p) = \min_{x' \in R_p} \{\sharp[x']_j\}$. The number $j(p)$ is the degree of inseparability of the extension $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$, that is $j(p) = i(p)$. In particular $i(p') = [\mathcal{H}(p') : \mathcal{H}(p)]/j(p)$.*

Proof. Let \tilde{C}_p and $\tilde{C}'_{p'}$ denote the smooth projective curves corresponding to the function fields $\widetilde{\mathcal{H}(p)}$ and $\widetilde{\mathcal{H}(p')}$ respectively. For a point $e \in \tilde{C}_p$ let s_e denote the uniformisat of the local ring $O_{\tilde{C}_p}$. The degree of inseparability of the extension $\widetilde{\mathcal{H}(p')}/\widetilde{\mathcal{H}(p)}$ is equal to $\min_{\{(e', e) \mid e' \in \tilde{C}'_{p'}, e' \mapsto e\}} \{\text{ord}_{e'}(s_e)\}$.

Let $x \in C(k)$ such that $\psi(r, x) = p$. Since $\Upsilon_{C^{\text{an}}}$ contains every k -point over which the morphism ϕ is ramified, ϕ is unramified over x . Let $e_x \in \tilde{C}_p$ correspond to the path $\psi(-, x) : [0, r] \rightarrow C^{\text{an}}$. Let t_x be a uniformisat of x such that $|t_x(p)| = 1$ and it does not have any zeros or poles at any y for which $\psi(-, y) : [0, r] \rightarrow C^{\text{an}} = e_x$.

It follows that the image of t_x in the field $\widetilde{\mathcal{H}(p)}$ is a uniformisat at the point e_x . Let $e' \in \tilde{C}'_{p'}$ which maps to e_x and $y' \in C'(k)$ such that the path $\psi'(-, y') : [0, r] \rightarrow C'^{\text{an}}$ coincides with e' in $T_{p'}$.

By the Non-archimedean Poincaré-Lelong Theorem, the order of vanishing of the uniformisat t_x at e' is equal to the cardinality of the equivalence class $[y]_j$. The equality $j(p) = \min_{x' \in R_p} \{\sharp[x']_j\}$ follows by taking into account the fact stated initially and the equality $i(p') = [\mathcal{H}(p') : \mathcal{H}(p)]/j(p)$ follows from the Lemma 4.1. \square

4.2. Calculating the ramification divisors $R_{p'}$. The ramification divisor $R_{p'}$ is defined as follows. As mentioned several times before, the morphism ϕ^{an} restricts to a surjective morphism between the skeleta $\Upsilon_{C'^{\text{an}}}$ and $\Upsilon_{C^{\text{an}}}$. Our choice of vertex sets $V(\Upsilon_{C'^{\text{an}}})$ and $V(\Upsilon_{C^{\text{an}}})$ was such that

$$V(\Upsilon_{C'^{\text{an}}}) = (\phi^{\text{an}})^{-1}(V(\Upsilon_{C^{\text{an}}})).$$

Let $p \in V(\Upsilon_{C^{\text{an}}})$ and $p' \in (\phi^{\text{an}})^{-1}(p)$. The morphism ϕ^{an} induces an extension of \tilde{k} -function fields $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$. Let \tilde{C}_p and $\tilde{C}'_{p'}$ denote the smooth projective \tilde{k} -curves associated to these function fields.

In the introduction of this section we decomposed the extension of \tilde{k} -function fields as follows. Let $I(p', p)$ denote the subfield of $\widetilde{\mathcal{H}(p')}$ which contains $\widetilde{\mathcal{H}(p)}$ such that $I(p', p)$ is purely inseparable over $\widetilde{\mathcal{H}(p)}$ of degree $i(p)$ and $I(p, p') \hookrightarrow \widetilde{\mathcal{H}(p)}$ is a separable field extension of degree $i(p')$. The function field $I(p', p)$ corresponds to a smooth projective \tilde{k} -curve which is homeomorphic to \tilde{C}_p . We denote this curve $W_{p', p}$.

The divisor $R_{p'}$ is the ramification divisor on the curve $\tilde{C}'_{p'}$ associated to the Riemann-Hurwitz formula for the finite separable \tilde{k} -morphism $\tilde{C}'_{p'} \rightarrow W_{p', p}$. It is defined as follows. Let $u' \in \tilde{C}'_{p'}$ and $u \in W_{p', p}$ be such that $u' \mapsto u$ and

$O_{\tilde{C}_{p'}, u'}$, $O_{W_{p', p}, u}$ be the local rings at the points u' and u respectively. The codifferent associated to the inclusion of local rings $O_{W_{p', p}, u} \hookrightarrow O_{\tilde{C}_{p'}, u'}$ is denoted $D(O_{\tilde{C}_{p'}, u'}, O_{W_{p', p}, u})$ and defined as follows.

$$D(O_{\tilde{C}_{p'}, u'}, O_{W_{p', p}, u}) := \{\beta \in \widetilde{\mathcal{H}(p')} \mid \text{Tr}_{\widetilde{\mathcal{H}(p')}/I_{p, p'}}(\beta O_{\tilde{C}_{p'}, u'}) \subseteq O_{W_{p', p}, u}\}.$$

The divisor $R_{p'}$ is such that

$$\text{ord}_{u'} R_{p'} := -\text{ord}_{u'}(D(O_{\tilde{C}_{p'}, u'}, O_{W_{p', p}, u})).$$

Our goal in this subsection is to study the divisor $R_{p'}$ using the compatible pair of deformation retractions ψ' and ψ constructed in section 3. We begin by using the deformation retractions ψ' and ψ constructed previously to study the local rings $O_{C_{p'}, u'}$ and $O_{W_{p', p}, u}$. Let $k(C')_{p'}^o := \{g \in k(C') \mid |g(p')| \leq 1\}$ and $k(C')_{p'}^{oo} := \{g \in k(C') \mid |g(p')| < 1\}$. We have the following equality

$$\widetilde{\mathcal{H}(p')} = k(C')_{p'}^o / k(C')_{p'}^{oo}.$$

Similarly,

$$\widetilde{\mathcal{H}(p)} = k(C)_p^o / k(C)_p^{oo}.$$

Let π denote the canonical surjections $k(C')_{p'}^o \rightarrow \widetilde{\mathcal{H}(p')}$ and $k(C)_p^o \rightarrow \widetilde{\mathcal{H}(p)}$. Let

$$A_{u'} := \{x' \in C'(k) \mid \psi'(-, x') : [0, 1] \rightarrow C'^{\text{an}} = u'\}$$

and

$$A_u := \{x \in C(k) \mid \psi(-, x) : [0, 1] \rightarrow C^{\text{an}} = u\}.$$

We now prove the following.

Lemma 4.3.

$$\pi((\cap_{x \in A_u} O_{C, x})_p^o) = O_{\tilde{C}_p, u}$$

and

$$\pi((\cap_{x' \in A_{u'}} O_{C, x'})_{p'}^o) = O_{\tilde{C}_{p'}, u'}.$$

Proof. We will only prove the first equality $\pi((\cap_{x \in A_u} O_{C, x})_p^o) = O_{\tilde{C}_p, u}$ and it will be clear that the same argument can be used to prove the second. The skeleta $\Upsilon_{C^{\text{an}}}$ is such that the open subspace $\{q \in C^{\text{an}} \mid \psi(-, q) : [0, 1] \rightarrow C^{\text{an}} = u\} \setminus \{p\}$ is isomorphic as an analytic space to the Berkovich open unit disk around 0. Using this it can be shown that for every point $x \in A_u$, there exists a uniformisat t_x of the local ring $O_{C, x}$ such that $|t_x(p)| = 1$, t_x doesnot have a zero or a pole at any point different from x in A_u and for any $y_1, y_2 \in A_u$, $\pi(t_{y_1}) = \pi(t_{y_2})$. As a result the ring $(\cap_{x \in A_u} O_{C, x})_p^o$ is a principal ideal domain.

Let $w \in k(C)_p^o$ be such that $\pi(w) \in O_{\tilde{C}_p, u}$ and is non zero. Let $\{x_1, \dots, x_m\}$ be the finite collection of points in A_u on which $r_i = \text{ord}_{x_i}(w)$ is non-zero. It follows that w is of the form $(\prod_i t_{x_i}^{r_i})w'$ where w' doesnot have a zero or pole at any point belonging to A_u and $|w'(p)| = 1$. Since $\pi(w) \in O_{\tilde{C}_p, u}$, by the non-archimedean Poincaré-Lelong theorem, we must have that $\sum_i r_i \geq 0$. It follows that $\pi(t_{x_1}^{\sum_i r_i} w') = \pi(w)$ and $\pi(t_{x_1}^{\sum_i r_i} w') = \pi(w)$. By definition, the element $t_{x_1}^{\sum_i r_i} w'$ belongs to $\cap_{x \in A_u} O_{C, x}$. We hence have the inequality $O_{\tilde{C}_p, u} \subset \pi((\cap_{x \in A_u} O_{C, x})_p^o)$. The inequality $\pi((\cap_{x \in A_u} O_{C, x})_p^o) \subseteq O_{\tilde{C}_p, u}$ is a consequence of the Poincaré-Lelong theorem. Hence the proof. \square

The preimage of the codifferent via the restriction $\pi|_{(\cap_{x' \in A_{u'}} O_{C, x'})_{p'}^o}$ will be a $(\cap_{x' \in A_{u'}} O_{C, x'})_{p'}^o$ -module. Extending scalars defines a module of the larger ring $\cap_{x' \in A_{u'}} O_{C, x'}$ which we call $T(u', u)$. The module $T(u', u)$ is such that.

$$\pi((T(u', u))_{p'}^o) = D(O_{\tilde{C}_{p'}, u'}, O_{W_{p', p}, u}).$$

When the extension $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$ is separable this ideal can be explicitly described as follows.

$$[T(u', u)]_{p'}^o := \{\beta \in k(C')_{p'}^o \mid [\text{Tr}_{\mathcal{H}(p')/\mathcal{H}(p)}(\beta(\cap_{x' \in A_{u'}} O_{C, x'})_{p'}^o)] \subseteq (\cap_{x \in A_u} O_{C, x})_p^o \bmod k(C)_p^{oo}\}$$

Proposition 4.4. *The following equality holds*

$$\text{ord}_{u'}(R_{p'}) = -\sum_{x' \in A_{u'}} \text{ord}_{x'} T(u', u).$$

Proof. The ring $\cap_{x' \in A_{u'}} O_{C', x'}$ is a principal ideal domain as discussed in the proof of Lemma 4.3. The $\cap_{x' \in A_{u'}} O_{C', x'}$ -module $T(u', u)$ is hence generated by an element of the form $(\prod_{x' \in A_{u'}} t_{x'}^{r_{x'}})w$ where $\{t_{x'}\}$ is the collection of uniformisants associated to every point of $A_{u'}$ introduced in the proof of 4.3, $r_{x'}$ is the order of vanishing of the ideal at the point x' and w is an element of the ring $k(C)_p^o$ which does not have a zero or pole at any point belonging to $A_{u'}$. A generator of this form is a generator of $T(u', u)_p^o$ as well.

It follows by the non-archimedean Poincaré-Lelong theorem, that

$$\sum_{x' \in A_{u'}} r_{x'} = \text{ord}_{u'} D(O_{\tilde{C}_{p'}, u'}, O_{W_{p', p}, u}) = -\text{ord}_{u'}(R_{p'})$$

Hence the proof. \square

When the morphism $\tilde{C}'_{p'} \rightarrow W_{p', p}$ is tamely ramified at the point u' , the order of vanishing of the ramification divisor can be calculated in terms of the deformation retractions as follows. Using Proposition 4.2 and (Theorem 4.16, [L]) the following can be easily demonstrated.

Proposition 4.5. *Let $x \in C(k)$ such that $\psi(1, x) = p$. Let r denote the cardinality of the set $\{x' \in C(k) \mid \phi(x') = x \text{ and } \psi(_, x') : [0, 1] \rightarrow C'^{\text{an}} = u'\}$. If the morphism $\tilde{C}'_{p'} \rightarrow W_{p', p}$ is tamely ramified at u' then $\text{ord}_{u'}(R_{p'}) = [r/(i(p))] - 1$.*

5. A SECOND CALCULATION OF THE GENUS $g(\Upsilon_{C^{\text{an}}})$.

As outlined in the introduction, in this section we introduce the divisor w on the skeleta $\Upsilon_{C^{\text{an}}}$ and relate the degree of this divisor to the genus of the skeleton $\Upsilon_{C'^{\text{an}}}$. The point of doing so is to study how $g(\Upsilon_{C^{\text{an}}})$ can be calculated in terms of the behaviour of the morphism between the sets of vertices.

We preserve our choices of vertex sets and edge sets for the two skeleta from the previous section. Let $p \in \Upsilon_{C^{\text{an}}}$ and n_p denote the number of preimages of p under the morphism ϕ . For any $p' \in C'^{\text{an}}$ such that $\phi(p') = p$, the morphism ϕ induces a map between the tangent spaces at p' and p (cf. Definition 2.6). Let E_p be the set of germs (cf. Section 2.4) for which there exists a representative starting from p and contained completely in $\Upsilon_{C^{\text{an}}}$. Hence E_p is a subset of the tangent space at p . For every $p' \in \phi^{-1}(p)$ and $e_p \in E_p$, let $l(e_p, p')$ denote the number of lifts of the germ e_p starting from p' . We define w as follows. Let $w(p) := (\sum_{e_p \in E_p, p' \in \phi^{-1}(p)} l(e_p, p')) - 2n_p$.

Proposition 5.1. *The degree of the divisor w is equal to $2g(\Upsilon_{C'^{\text{an}}}) - 2$.*

Proof. We begin by stating the following fact about connected, finite graphs. Let Γ be homeomorphic to a connected, finite graph and let $p \in \Gamma$. Let t_p denote the number of distinct germs of paths starting at p and contained in Γ . We set

$D_\Gamma := \sum_{p \in \Gamma} (t_p - 2) \cdot p$. Observe that D_Γ is a divisor on the finite graph Γ whose degree is equal to $2g(\Gamma) - 2$.

The connected, finite graphs $\Upsilon_{C'^{\text{an}}}$ and $\Upsilon_{C^{\text{an}}}$ are the images of a pair of compatible deformation retractions. Hence the morphism ϕ^{an} restricts to a continuous map $\Upsilon_{C'^{\text{an}}} \rightarrow \Upsilon_{C^{\text{an}}}$. This map induces a homomorphism $\phi_* : \text{Div}(\Upsilon_{C'^{\text{an}}}) \rightarrow \text{Div}(\Upsilon_{C^{\text{an}}})$ defined as follows. We define ϕ_* only on the generators of the group $\text{Div}(\Upsilon_{C'^{\text{an}}})$. If $1 \cdot p' \in \text{Div}(\Upsilon_{C'^{\text{an}}})$ then we set $\phi_*(1 \cdot p') = 1 \cdot \phi(p')$. Note that for any divisor $D' \in \text{Div}(\Upsilon_{C'^{\text{an}}})$, $\deg(\phi_*(D')) = \deg(D')$.

We will show that $w = \phi_*(D_{\Upsilon_{C'^{\text{an}}}})$. By definition,

$$\phi_*(D_{\Upsilon_{C'^{\text{an}}}})(p) = \left(\sum_{p' \in \phi^{-1}(p)} t_{p'} \right) - 2 \cdot n_p.$$

For a point $p' \in \phi^{-1}(p)$, every germ for which there exists a representative contained in $(\Upsilon_{C'^{\text{an}}})$ and starting from p' is the lift of a germ for which there exists a representative starting from p and contained in $(\Upsilon_{C^{\text{an}}})$. It follows that $\sum_{p' \in \phi^{-1}(p)} t_{p'} = \sum_{e_p \in E_p, p' \in \phi^{-1}(p)} l(e_p, p')$. Hence $\phi_*(D_{\Upsilon_{C'^{\text{an}}}}) = w$. \square

5.1. Calculating n_p for $p \in C^{\text{an}}$ of type I or II when the extension of function fields $k(C) \hookrightarrow k(C')$ is Galois. In this section we impose the added restriction that the extension of function fields $k(C) \hookrightarrow k(C')$ associated to the morphism ϕ is Galois.

Let $x \in C(k)$ and $x' \in C'(k)$ such that $\phi(x') = x$. Let $\text{ram}(x', x)$ denote the ramification degree associated to the extension of the discrete valuation rings $O_{C, x} \hookrightarrow O_{C', x'}$. Since the morphism ϕ is Galois, for $x \in C(k)$, the ramification degree $\text{ram}(x', x)$ is a constant as x' varies along the set of preimages of the point x . The ramification degree depends only on the point $x \in C(k)$ and we denote it $\text{ram}(x)$. As k is algebraically closed we have that

$$[k(C') : k(C)] = n_x \text{ram}(x)$$

where n_x is cardinality of the fiber over x . We may hence restrict our attention to points of type II.

If p is a point of type II then let s_p be the smallest real number in the real interval $[0, 1]$ such that p belongs to $\psi(s_p, C^{\text{an}})$ where ψ is the deformation retraction constructed in section 3. Since the pair of deformation retractions constructed in section 3 were compatible, we must have that $(\phi^{\text{an}})^{-1}(p) \subset \psi'(s_p, C'^{\text{an}})$.

We define an equivalence relation $\sim_{r(s_p)}$ on the set of k -points of the curve C' . Let $x'_1, x'_2 \in C'(k)$. We set $x'_1 \sim_{r(s_p)} x'_2$ if $\phi(x'_1) = \phi(x'_2)$ and $\psi'(s_p, x'_1) = \psi'(s_p, x'_2)$. Observe that each equivalence class is finite. For $x' \in C'(k)$, let $[x']_{r(s_p)}$ denote that equivalence class containing the point x' .

Lemma 5.2. *If $x'_1, x'_2 \in C'(k)$ such that $\phi(x'_1) = \phi(x'_2)$ then*

$$\# [x'_1]_{r(s_p)} = \# [x'_2]_{r(s_p)}.$$

Proof. The lemma is true if $x'_1 \sim_{r(s_p)} x'_2$. Let us hence assume that $\psi'(s_p, x'_1) = p'_1$ and $\psi'(s_p, x'_2) = p'_2$ where p'_1 and p'_2 are two points on C'^{an} . Observe that since $\Upsilon_{C'^{\text{an}}}$ and $\Upsilon_{C^{\text{an}}}$ are the images of a pair of compatible deformation retractions, $\phi^{\text{an}}(p'_1) = \phi^{\text{an}}(p'_2)$. Let $p := \phi^{\text{an}}(p'_1)$. The Galois group $G := \text{Gal}(k(C')/k(C))$ acts transitively on the set of preimages $\phi^{-1}(p)$. Let $\sigma \in G$ be an element of the Galois group such that $\sigma(p'_1) = p'_2$. By proposition 3.7 if $a \sim_{r(s_p)} x'_1$, then $\sigma(a) \sim_{r(s_p)} x'_2$. As σ is bijective, $\# [x'_1]_{r(s_p)} \leq \# [x'_2]_{r(s_p)}$. By symmetry we conclude that the lemma is true. \square

Let $x \in C(k)$ and $x' \in C'(k)$ such that $\phi(x') = x$. The lemma above implies that the cardinality of the equivalence class $[x']_{r(s_p)}$ depends only on x . Hence we set

$$r_{s_p}(x') := \sharp[x']_{r(s_p)}.$$

Since the extension $k(C) \hookrightarrow k(C')$ is Galois, the degree of the field extensions $[\mathcal{H}(p') : \mathcal{H}(p)]$ remains constant as p' varies along the set of preimages of the point p . As this value depends only on p we denote it $f(p)$.

Proposition 5.3. *Let $p \in \Upsilon_{C^{\text{an}}}$ and $x \in C(k)$ such that $\psi(s_p, x) = p$. We have the following equality*

$$n_p = [k(C') : k(C)] / r_{s_p}(x).$$

Proof. Given a point $p \in C^{\text{an}}$, we have that

$$[k(C') : k(C)] = n_p f(p)$$

Let $p' \in (\phi^{\text{an}})^{-1}(p)$ and $x' \in C'(k)$ such that $\psi'(s_p, x') = p'$. We will use $\tilde{C}_{p'}$ and \tilde{C}_p to denote the smooth projective \tilde{k} -curves which correspond to the function fields $\widetilde{\mathcal{H}(p')}$ and $\widetilde{\mathcal{H}(p)}$. By Lemma 4.1, we have that

$$f(p) = [\widetilde{\mathcal{H}(p')} : \widetilde{\mathcal{H}(p)}].$$

The rest of the proof uses arguments similar to those employed to prove Lemma 4.1. Let t_x be a uniformisat of the local ring $O_{C,x}$ such that $|t_x(p)| = 1$ and t_x does not have a zero or a pole at any point $y \in C(k)$ for which the path $\psi(_, y) : [0, s_p] \rightarrow C^{\text{an}}$ coincides with the path $e_x := \psi(_, x) : [0, s_p] \rightarrow C^{\text{an}}$ in the tangent space T_p . Let \tilde{t}_x denote the image of t_x in $\widetilde{\mathcal{H}(p)}$ and t'_x its image in $k(C')$.

Let $E(e_x, p')$ denote the set of preimages of the point $e_x \in \tilde{C}_p$ on $\tilde{C}_{p'}$. The set $E(e_x, p')$ coincides with the collection of paths $\psi(_, y) : [0, s_p] \rightarrow C^{\text{an}}$ as y varies along the elements of the equivalence class $[x']_{r(s_p)}$. We have that

$$[\widetilde{\mathcal{H}(p')} : \widetilde{\mathcal{H}(p)}] = \sum_{e' \in E(e_x, p')} \text{ord}_{e'}(\tilde{t}_x).$$

Our choice of t_x implies that t'_x cannot have a zero or pole at any point $y' \in C'(k)$ for which $\psi'(_, y') : [0, r] \rightarrow C'^{\text{an}} \in E(e_x)$ and $y' \notin \phi^{-1}(x)$. Since the skeleton $\Upsilon_{C^{\text{an}}}$ contains the set of k -points belonging to C over which the morphism ϕ is ramified, we must have that $\text{ram}(x) = 1$.

It follows from the Poincaré-Lelong theorem that

$$\sum_{e' \in E(e_x, p')} \text{ord}_{e'}(\tilde{t}_x) = r_{s_p}(x).$$

Hence the proof. \square

5.1.1. Calculating $l(e_p, p')$ for $p \in \Upsilon_{C^{\text{an}}}$ and $p' \in \phi^{-1}(p)$. In this subsection we study the number $l(e_p, p')$ for $p \in \Upsilon_{C^{\text{an}}}$ and $p' \in \phi^{-1}(p)$ using arguments similar to those employed in the proof of 5.3. In what follows we will assume that p is a type II point.

Lemma 5.4. *Let $p \in C^{\text{an}}$ and $e_p \in E_p$, then $l(e_p, p')$ is a constant as p' varies through the set of preimages of the morphism ϕ .*

Proof. Let $p'_1, p'_2 \in \phi^{-1}(p)$. The Galois group $\text{Gal}(k(C')/k(C))$ acts transitively on the set of preimages of the point p . The elements of the Galois group are continuous homeomorphisms on C'^{an} . If $\sigma \in \text{Gal}(k(C')/k(C))$ is such that $\sigma(p'_1) = p'_2$, then σ maps the set of germs $l(e_p, p'_1)$ injectively to the set $l(e_p, p'_2)$. By symmetry we may conclude that our proof is complete. \square

By the lemma above, if $p \in \Upsilon_{C^{\text{an}}}$ and $e_p \in E_p$ then $l(e_p) := l(e_p, p')$ for any $p' \in \phi^{-1}(p)$ is well defined. In section 2 we discussed the tangent space at a point on the space C^{an} . The germ of e_p is an element of the tangent space at p (cf. Definition 2.6). By Section 2.4.3, e_p corresponds to a discrete valuation of the \tilde{k} -function field $\widetilde{\mathcal{H}(p)}$. For any $p' \in \phi^{-1}(p)$, the extension of fields $\widetilde{\mathcal{H}(p)} \hookrightarrow \widetilde{\mathcal{H}(p')}$ can be decomposed into the composite of a purely inseparable extension and a Galois extension. Hence the ramification degree $\text{ram}(e'/e_p)$ is constant as e' varies through the set of lifts of the germ of e_p . Let $\text{ram}(e_p)$ be this number. We have that

Proposition 5.5.

$$l(e_p, p') = ([k(C') : k(C)] / (n_p \cdot \text{ram}(e_p)))$$

Proof. The morphism $\phi : C' \rightarrow C$ corresponds to an extension of function fields $k(C) \rightarrow k(C')$ which is Galois. The point $p \in \Upsilon_{C^{\text{an}}}$ induces a multiplicative norm on the function field $k(C)$. The set of preimages $\phi^{-1}(p)$ corresponds to those multiplicative norms on $k(C')$ which extend the multiplicative norm p on $k(C)$. For every $p' \in \phi^{-1}(p)$, $\mathcal{H}(p')$ is the completion of $k(C')$ for p' and is a finite extension of the non-archimedean valued complete field $\mathcal{H}(p)$. The Galois group $\text{Gal}(k(C')/k(C))$ acts transitively on the set $\phi^{-1}(p)$. It follows that degree of the extension $[\mathcal{H}(p') : \mathcal{H}(p)]$ is a constant as p' varies through the set $\phi^{-1}(p)$. We denoted this number $f(p)$. Hence we have that

$$[k(C') : k(C)] = n_p f(p).$$

By Lemma 4.1, $f(p) = [\widetilde{\mathcal{H}(p')} : \widetilde{\mathcal{H}(p)}]$. Uniquely associated to the \tilde{k} -function fields $\mathcal{H}(p)$ and $\mathcal{H}(p')$ are smooth, projective \tilde{k} -curves denoted \tilde{C}_p and $\tilde{C}_{p'}$. The germ e_p corresponds to a closed point on the former of these curves. The number $l(e_p, p')$ is the cardinality of the set of preimages of the closed point e_p for the morphism induced by ϕ^{an} . The result now follows from (Theorem 7.2.18, [L]) applied to the \tilde{k} -function fields $\widetilde{\mathcal{H}(p)}$, $\widetilde{\mathcal{H}(p')}$ and the divisor e_p . □

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